TAMENESS FOR CONNECTED COMPONENTS OF SOME SUBSETS OF BERKOVICH SPACES

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Abstract. Let k be a non-Archimedean field which is discretely and non trivially valued. We prove that the complementary set of the image of a morphism between strictly k-affinoid spaces has finitely many connected components and that this property holds more generally for finite Boolean combinations of such images. This happens to be a consequence of a more general result for subanalytic sets (as they have been defined by Leonard Lipshitz): we show that if S is a subanalytic set of some strictly k-affinoid space, then it has finitely many connected components.

Contents

1. Introduction

Let k be a non-Archimedean field which is discretely and non-trivially valued. In non-Archimedean geometry one of the advantages of Berkovich spaces, also called k-analytic spaces [Ber90, Ber93, Duc07], is to offer some genuine topological spaces, with good topological properties (locally arcwise-connected, locally compact, etc) which is not possible in the formalism of rigid spaces (which are equipped with a Grothendieck topology instead [BGR84]) or in the formalism of formal schemes.

Building blocks of k-analytic spaces are k-affinoid spaces. It is well known that k-affinoid spaces have finitely many connected components, for instance because they are compact and locally connected spaces. As a consequence, if $f : Y \to X$ is a morphism of k-affinoid spaces, $f(Y) \subset X$ has finitely many connected components.

But let us consider a morphism of strictly k-affinoid spaces $f : Y \to X$. One could wonder if the complementary set $f(Y)^c$ has finitely many connected components. We do not see any obvious reason for this: for a general X, there are many compact sets $S \subset X$ whose complementary set has infinitely many connected components. For instance, if one takes $X = \mathbb{B}$ the closed unit disc and $S = \{\eta\}$ where η is the Gauss point of $\mathbb{B}$, then $X \setminus \{\eta\}$ has infinitely many connected components. Though,

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the answer the the above question is yes, and we prove more generally the following result.

Proposition 1.1. *Let X be a strictly k-affinoid space. Let $S \subset X$ be a finite Boolean combination of subsets of the form $f(Y)$ where $f : Y \to X$ is a morphism of strictly k-affinoid spaces. Then S has finitely many connected components and each of them is path-connected.*

In fact this is just a particular case of the following result. A special k°-algebra is a quotient of some $k^\circ\langle X_1, \dots, X_m\rangle[[\rho_1, \dots, \rho_n]]$.

Theorem 1.2. *Let A be a special k°-algebra, $\mathfrak{X} := Spf(A)$ and let $S \subset \mathfrak{X}_\eta$ be a subanalytic set. Then S has finitely many connected components and each of them is path-connected.*

The proof of Theorem 1.2 relies on two results results.

The first one is the quantifier elimination theorem for subanalytic sets due to Leonard Lipshitz and Zachary Robinson [Lip93, Theorem 3.8.1] and [LR00a, Corollary 4.3]. However, in order to use this theorem, we have to explain [Mar13, 3.1.1] why subanalytic sets make perfect sense in the context of Berkovich spaces.

The second result is that if A is a special k°-algebra which is a domain, and $\mathfrak{X} = \mathrm{Spf}(A)$ is the associated formal scheme over $\mathrm{Spf}(k^\circ)$, then $\mathfrak{X}_\eta$, the k-analytic space attached to $\mathfrak{X}$, is irreducible. This result is due to Brian Conrad [Con99, Theorem 2.3.1] and relies itself on a theorem due to Aise Johan de Jong [dJ95, Theorem 7.4.1].

Remark 1.3. Theorem 1.2 has strong connections with Antoine Ducros result [Duc03, Theorem 3.2], which asserts that semi-algebraic sets have finitely many connected components. In particular, if we assume that X, Y and f are defined by polynomials, Proposition 1.1 is a consequence of Ducros result.

Let us also mention the recent work of Ehud Hrushovski and Françcois Loeser [HL10] which implies much stronger topological tameness properties for k-analytic spaces defined by algebraic conditions.

Organization of the paper. In section 2 we recall the main results concerning subanalytic sets that we will need. In particular, in Remark 2.14 we explain why Theorem 1.2 implies Proposition 1.1. In section 3 we prove Theorem 1.2.

Notations. If K is a non-Archimedean field we will set

$$K^\circ = \{x \in K \mid |x| \le 1\}.$$

$$K^{\circ\circ} = \{x \in K \mid |x| < 1\}.$$

In this text, k-analytic space refers to the spaces defined by Vladimir Berkovich in [Ber90, Ber93]. We will denote by $\mathbb{B}$ the closed unit disc over k (seen as a k-analytic space), and $\mathbb{D}$ the open unit disc over k (seen as a k-analytic space).

2. Subanalytic sets

2.1. **Special formal schemes.** Let $k^\circ\langle X_1, \dots, X_m\rangle$ be the set of power series $\sum_{\nu\in\mathbb{N}^n} a_\nu X^\nu$ such that $a_\nu \in k^\circ$ and $|a_\nu| \to 0$. Let π be some uniformizer of k. We consider $k^\circ\langle X_1, \dots, X_m\rangle[[\rho_1, \dots, \rho_n]]$ as an adic algebra, with $(\pi, \rho_1, \dots, \rho_n)$ as an ideal of definition.

Definition 2.1. [Ber96a, section 1] A special k°-algebra is an adic algebra which is a quotient of some $k^\circ\langle X_1,\dots,X_m\rangle[[\rho_1,\dots,\rho_n]]$. In that case, we call $\mathfrak{X}=\mathrm{Spf}(A)$ a special formal scheme.

Let A be a special k°-algebra, and let $\mathfrak{X}:=\mathrm{Spf}(A)$, the $\mathrm{Spf}(k^\circ)$-formal scheme associated to A. We denote by $\mathfrak{X}_\eta$ the k-analytic space associated to $\mathfrak{X}$ (see [Ber96a, Section 1]). If $A=k^\circ\langle X_1,\dots,X_m\rangle[[\rho_1,\dots,\rho_n]]/I$ where I is an ideal of A, by definition, $\mathfrak{X}_\eta$ is isomorphic to the closed analytic subset of $\mathbb{B}^m\times\mathbb{D}^n$ defined by the ideal I, that is to say, $\mathfrak{X}_\eta=\{x\in\mathbb{B}^m\times\mathbb{D}^n \mid f(x)=0 \ \ \forall f\in I\}$. For instance,

$$\mathrm{Spf}(k^\circ\langle X\rangle)_\eta\simeq\mathbb{B} \quad\text{and}\quad \mathrm{Spf}(k^\circ[[\rho]])_\eta\simeq\mathbb{D}.$$

More generally,

$$\mathrm{Spf}\big(k^\circ\langle X_1,\dots,X_m\rangle[[\rho_1,\dots,\rho_n]]\big)_\eta\simeq\mathbb{B}^m\times\mathbb{D}^n.$$

Beware that the product $\times$ above is intended as a product in the category of k-analytic spaces. In particular, $\mathbb{B}^m\times\mathbb{D}^n$ is not the set theoretical product of $\mathbb{B}^m$ and $\mathbb{D}^n$. Note that $\mathfrak{X}_\eta$ can be defined in an more intrinsic way, and is independent of the presentation of A as $k^\circ\langle X_1,\dots,X_m\rangle[[\rho_1,\dots,\rho_n]]/I$. See [dJ95, 7.1] or [Ber96b, 0.2.6] for more on this.

Definition 2.2. Let A be a special k°-algebra, and let $\mathfrak{X}=\mathrm{Spf}(A)$.

(1) Let $I\subset A$ be some ideal. We set

$$V(I)=\{x\in\mathfrak{X}_\eta \mid f(x)=0\ \forall f\in I\}.$$

$V(I)$ is a closed analytic set of $\mathfrak{X}_\eta$.

(2) We say that a set $Z\subset\mathfrak{X}_\eta$ is constructible if it is a finite Boolean combination of some $V(I)$'s.

Remark 2.3. (1) There is a natural morphism of k-algebras

$$k\otimes_{k^\circ}A\to\Gamma(\mathfrak{X}_\eta,\mathcal{O}_{\mathfrak{X}_\eta}).$$

Its image clearly goes to the set of bounded analytic functions on $\mathfrak{X}_\eta$. So in general, this morphism is far from being surjective precisely because in general there exist some analytic functions on $\mathfrak{X}_\eta$ which are unbounded. But if A is a quotient of $k^\circ\langle X_1,\dots,X_m\rangle$, this is an isomorphism.

(2) If A is normal, A.J. de Jong has proved [dJ95, 7.4.1] that with the above morphism, A identifies with the set of analytic functions f on $\mathfrak{X}_\eta$ such that $|f(x)|\le 1$ for all $x\in\mathfrak{X}_\eta$.

(3) De Jong's result actually implies that if A is reduced, $k\otimes_{k^\circ}A$ is isomorphic to the sets of bounded analytic functions on $\mathfrak{X}_\eta$ (see [Mar13, 0.5.3] for a proof of this result which has been communicated to us by Christian Kappen).

2.2. **Analytic Quantifier elimination Theorem.** We assume some basic knowledge of model theory: languages, terms, structures and quantifier elimination. See for instance [Mar00, Chapter 1]

The language. We define L^D_{an}, a three-sorted language. Its sorts are: $\mathcal{O},\mathfrak{m}$ and Γ_0. On $\mathcal{O}$ and $\mathfrak{m}$ there are function symbols $+,-,\cdot$ and there is a function symbol $\cdot$ on Γ_0. On Γ_0, there is a relation symbol $<$ and a constant 0. There are function symbols $D_0:\mathcal{O}^2\to\mathcal{O}$, $D_1:\mathcal{O}^2\to\mathfrak{m}$, $|\cdot|:\mathcal{O}\to\Gamma_0$ and for each $f\in k^\circ\langle X_1,\dots,X_m\rangle[[\rho_1,\dots,\rho_n]]$, there is a function symbol $f:\mathcal{O}^m\times\mathfrak{m}^n\to\mathcal{O}$.

If in addition $f \in (\pi,\rho)k^\circ\langle X_1,\ldots,X_m\rangle[[\rho_1,\ldots,\rho_n]]$, there is a function symbol $f : \mathcal{O}^m \times \mathfrak{m}^n \to \mathfrak{m}$. By definition of a term in a language, an L_{an}^D-term is everything that can be built using composition of these symbols.

The standard structures. Let $k \to K$ be some non-Archimedean extension. We can associate to K a *standard* L_{an}^D-structure as follows. $(\mathcal{O},+,-,.)$ is interpreted as $(K^\circ,+,-,.)$, $(\mathfrak{m},+,-,.)$ is interpreted as $(K^{\circ\circ},+,-,.)$ and $(\Gamma_0,<,.)$ is interpreted as $(|K|,<,.)$. Symbol functions $f \in k^\circ\langle X_1,\ldots,X_m\rangle[[\rho_1,\ldots,\rho_n]]$ are naturally interpreted as functions $f : (K^\circ)^m \times (K^{\circ\circ})^n \to K^\circ$. Indeed, f is a power series which converges on $(K^\circ)^m \times (K^{\circ\circ})^n$ and its norm is bounded by 1. Finally, D_0 and D_1 are interpreted as follows:

$$\begin{array}{rccc} D_0 : & (K^\circ)^2 & \to & K^\circ \\ & (x,y) & \mapsto & \begin{cases} \frac{x}{y} & \text{if } |x| \leq |y| \neq 0 \\ 0 & \text{otherwise} \end{cases} \\ D_1 : & (K^\circ)^2 & \to & K^{\circ\circ} \\ & (x,y) & \mapsto & \begin{cases} \frac{x}{y} & \text{if } |x| < |y| \\ 0 & \text{otherwise} \end{cases}. \end{array}$$

For short we will denote this structure by K instead of $(K^\circ, K^{\circ\circ}, |K|)$.

Theorem 2.4. [Lip93, 3.8.1] *and* [LR00a, 4.3]. *Let φ be a first order L_{an}^D-formula. There exists ψ a quantifier free L_{an}^D-formula such that for all algebraically closed non-Archimedean extension $k \to K$*

$$K \models (\varphi \Leftrightarrow \psi).$$

Remark 2.5.

(1) Another way to state the above theorem is to say that if $k \to K$ is an algebraically closed non-Archimedean extension, the standard structure associated to K has quantifier elimination, and moreover that if $k \to K \to L$ are two algebraically closed non-Archimedean extensions, then $(K^\circ, K^{\circ\circ}, |K|) \to (L^\circ, L^{\circ\circ}, |L|)$ is an elementary extension, in the sense of model theory (see [Mar00, 2.3.1]).

(2) In [LR00a, 4.3], the above result is in fact stated for any non-Archimedean field k ($k = \mathbb{C}_p$ for instance), in particular not necessarily discretely valued. In this general setting, the above formalism has to be changed. For instance, if $k = \mathbb{C}_p$, one should note consider the rings $\mathbb{C}_p^\circ\langle X_1,\ldots,X_m\rangle[[\rho_1,\ldots,\rho_n]]$ which are not tame enough. Indeed, as explained in [LR00b, p.6-7], there are some power series $f \in \mathbb{C}_p^\circ[[\rho_1]]$ whose vanishing locus in $\mathbb{D}$ is an infinite countable discrete set, which thereby will have infinetely many connected componnents. So when k is not necessarily discretely valued, one has to introduce some news rings called $S_{m,n}^\circ(E,k)$(see [LR00b, 2.1.1]), where k is our non-Archimedean base field, and $E \subset k$ is a discrete valuation subring of k. These rings $S_{m,n}^\circ(E,k)$ will play the role of $k^\circ\langle X_1,\ldots,X_m\rangle[[\rho_1,\ldots,\rho_n]]$ in the above theorem.

(3) According to [LR00b, Theorem 2.1.3 (i)], when k is dicretely valued, $S_{m,n}^\circ(k^\circ,k) = k^\circ\langle X_1,\ldots,X_m\rangle[[\rho_1,\ldots,\rho_n]]$. Consequently, the above theorem is just an instance of [LR00a, Corollary 4.3] for the choice $S_{m,n}(k^\circ,k)$.

2.3. **Subanalytic sets in the Berkovich setting.** In this section, we explain how we can make sense of subanalytic sets in Berkovich poydiscs, and how the quantifier elimination Theorem 2.4 can be interpreted in this context.

Definition 2.6. Let $k \to K$ be some non-Archimedean extension and $n \in \mathbb{N}$. We associate to it a natural map $p_K : K^n \to \mathbb{A}_k^{n,\mathrm{an}}$. It sends a point $a \in K^n$ to the multiplicative seminorm

$$P \in k[X_1, \dots, X_n] \mapsto |P(a)|_K$$

where $|\cdot|_K$ refers to the norm on K. In our context, if $m \in \mathbb{N}$, we will also denote by p_K the induced map $p_K : (K^\circ)^m \times (K^{\circ\circ})^n \to \mathbb{B}^m \times \mathbb{D}^n$. If m, m' are integers, we will use freely that the following diagram commutes

$$\begin{tikzcd} K^{m+m'} \arrow[r, "p_K"] \arrow[d, "pr"] & \mathbb{A}_k^{m+m',\mathrm{an}} \arrow[d, "pr"] \\ K^m \arrow[r, "p_K"] & \mathbb{A}_k^{m,\mathrm{an}} \end{tikzcd}$$

where $pr : K^{m+m'} \to K$ and $pr : \mathbb{A}_k^{m+m',\mathrm{an}} \to \mathbb{A}_k^{m,\mathrm{an}}$ are the natural coordinate projections (we should have given to them two different names).

Lemma 2.7. *Let $f : \mathcal{O}^m \times \mathfrak{m}^n \to \mathcal{O}$ be an L_{an}^D-term. Let $z \in \mathbb{B}^m \times \mathbb{D}^n$. Let $k \to K$ and $k \to L$ be two non-Archimedean extensions and let $a \in (K^\circ)^m \times (K^{\circ\circ})^n$, $b \in (L^\circ)^m \times (L^{\circ\circ})^n$ such that $p_K(a) = z = p_L(b)$. Then*

$$|f(a)| = |f(b)|$$

where the left (resp. right) hand-side is interpreted in the L_{an}^D-structure K (resp. L).

Proof. The coordinates functions on $\mathbb{B}^m \times \mathbb{D}^n$ allow us to associate to $z \in \mathbb{B}^m \times \mathbb{D}^n$ a tuple $\underline{z} \in (\mathcal{H}(z)^\circ)^m \times (\mathcal{H}(z)^{\circ\circ})^n$. By the definition of the completed residue fields, there are unique non-Archimedean extensions

$$\begin{tikzcd} & K \\ \mathcal{H}(z) \arrow[ur, "\iota_K"] \arrow[dr, "\iota_L"'] & \\ & L \end{tikzcd}$$

such that ι_K sends $\underline{z}$ to a and ι_L sends $\underline{z}$ to b. This implies that $|f(\underline{z})| = |f(a)|$ and that $|f(\underline{z})| = |f(b)|$. $\square$

Definition 2.8. Let $f : \mathcal{O}^m \times \mathfrak{m}^n \to \mathcal{O}$ be an L_{an}^D-term and $z \in \mathbb{B}^m \times \mathbb{D}^n$. We denote by $|f(z)|$ the real number defined above.

Definition 2.9. A set $S \subset \mathbb{B}^m \times \mathbb{D}^n$ is subanalytic if it is a finite Boolean combination of sets of the form $\{x \in \mathbb{B}^m \times \mathbb{D}^n \mid |f(x)| \leq |g(x)|\}$ where $f, g : \mathcal{O}^m \times \mathfrak{m}^n \to \mathcal{O}$ are L_{an}^D-terms.

Remark 2.10. By definition of the language L_{an}^D, a quantifier free L_{an}^D-formula $\varphi(x_1, \dots, x_m, \rho_1, \dots, \rho_n)$ is a finite Boolean combination of formulas $|f| \leq |g|$ where $f, g : \mathcal{O}^m \times \mathfrak{m}^n \to \mathcal{O}$ are L_{an}^D-terms. Hence, to such formula, one can associate a subanalytic set. If one wants to, one can say that we are facing a homomorphism of Boolean algebras.

Remark 2.11. Let $f, g : \mathcal{O}^m \times \mathfrak{m}^n \to \mathcal{O}$ be some L_{an}^D-terms. The subsets of $\mathbb{B}^m \times \mathbb{D}^n$ defined by the conditions $\{|f| < |g|\}, \{|f| = |g|\}, \{f \neq 0\}, \{f = 0\}$ are subanalytic. For the first one, take the complementary set of $\{|g| \leq |f|\}$, and for the last one, remark that $\{f = 0\} = \{|f| \leq |0|\}$. Moreover, if X is a k-affinoid space defined as the vanishing locus in $\mathbb{B}^m$ of some functions $f_1, \dots, f_N \in k^\circ\langle X_1, \dots, X_m\rangle$, then X corresponds to the subanalytic set of $\mathbb{B}^n$

$$S = \bigwedge_{i=1\dots N} \{f_i = 0\}.$$

Lemma 2.12. *Let $S \subset \mathbb{B}^{m+m'} \times \mathbb{D}^{n+n'}$ be a subanalytic set and let*

$$pr : \mathbb{B}^{m+m'} \times \mathbb{D}^{n+n'} \to \mathbb{B}^m \times \mathbb{D}^n$$

be the coordinate projection. Then $pr(S)$ is a subanalytic set of $\mathbb{B}^m \times \mathbb{D}^n$.

Proof. Let $\varphi = \varphi(x_1, \dots, x_{m+m'}, \rho_1, \dots, \rho_{n+n'})$ be some quantifier free L_{an}^D-formula defining S as in Remark 2.10. Let $\psi = \psi(x_1, \dots, x_m, \rho_1, \dots, \rho_n)$ be some quantifier free L_{an}^D-formula equivalent, in the sense of Theorem 2.4, to the formula

$$\exists x_{m+1}, \dots, x_{m+m'} \in \mathcal{O}\ \exists \rho_{n+1}, \dots \rho_{n+n'} \in \mathfrak{m}.\ \varphi(x_1, \dots, x_{m+m'}, \rho_1, \dots, \rho_{n+n'}).$$

Let $T \subset \mathbb{B}^m \times \mathbb{D}^n$ be the subanalytic set attached to ψ as in Remark 2.10. We claim that $T = pr(S)$.

$pr(S) \subset T$. Let $x \in pr(S)$ and let $y \in S$ be some preimage of x. Let $k \to K$ be some non-Archimedean extension and $b \in (K^\circ)^{m+m'} \times (K^{\circ\circ})^{n+n'}$ such that $p_K(b) = y$. It follows that $K \models \varphi(b)$. Let then $a \in (K^\circ)^m \times (K^{\circ\circ})^n$ be the projection of b on $(K^\circ)^m \times (K^{\circ\circ})^n$. Then $p_K(a) = x$. In addition, ψ, $K \models \psi(a)$. Hence by definition of ψ, $K \models \varphi(b)$, and $x \in T$. Here is a diagram with the involved elements.

$$\begin{tikzcd} b \in (K^\circ)^{m+m'} \times (K^{\circ\circ})^{n+n'} \arrow[r, "p_K"] \arrow[d, "pr"] & \mathbb{B}^{m+m'} \times \mathbb{D}^{n+n'} \supset S \ni y \arrow[d, "pr"] \\ a \in (K^\circ)^m \times (K^{\circ\circ})^n \arrow[r, "p_K"] & \mathbb{B}^m \times \mathbb{D}^n \ni x \end{tikzcd}$$

$T \subset pr(S)$. Let $x \in T$. Let $k \to K$ be some non-Archimedean extension and let $a \in (K^\circ)^m \times (K^{\circ\circ})^n$ such that $p_K(a) = x$. Then (still using Lemma 2.7) $K \models \varphi(a)$. Hence by definition of φ, there exists some $b \in (K^\circ)^{m+m'} \times (K^{\circ\circ})^{n+n'}$ which projects to $a \in (K^\circ)^m \times (K^{\circ\circ})^n$ such that $K \models \psi(b)$. So if $y := p_K(b)$, then $y \in S$. In addition, $p(y) = x$. $\square$

Definition 2.13. Let $\mathfrak{X} = \mathrm{Spf}(A)$ be a special formal scheme where $A = k^\circ\langle X_1, \dots, X_m\rangle[[\rho_1, \dots, \rho_n]]/I$, and let us consider the associated embedding $\mathfrak{X}_\eta \subset \mathbb{B}^m \times \mathbb{D}^n$. Then $\mathfrak{X}_\eta$ is a subanalytic set of $\mathbb{B}^m \times \mathbb{D}^n$. We say that $S \subset \mathfrak{X}_\eta$ is a subanalytic set of $\mathfrak{X}_\eta$ when it is subanalytic in $\mathbb{B}^m \times \mathbb{D}^n$. In particular, if X is a k-affinoid space, we can see it as some $\mathfrak{X}_\eta$, and we say that $S \subset X$ is subanalytic if it fulfills the above condition.

It is easy to check that this definition does not depend on the chosen presentation of A.

Remark 2.14. According to the above Lemma 2.12, and remark 2.11, we know that the image of a morphism of k-affinoid spaces can be described as a subanalytic set. Indeed, if $X \subset \mathbb{B}^m$ and $Y \subset \mathbb{B}^{m'}$ are affinoid spaces and $f : X \to Y$ is a morphism of affinoid spaces, then the graph of f, is a subanalytic set of $\mathbb{B}^{m+m'}$ and its projection on the last m' coordinates is then a subanalytic set. So Theorem 1.2 implies Proposition 1.1 because the class of subanalytic sets is by definition stable under complement and because any k-affinoid space is isomorphic to some $\mathfrak{X}_\eta$.

Remark 2.15. More generally, let $\mathfrak{X}$ be a locally Noetherian $\mathrm{Spf}(k^\circ)$ adic formal scheme spanned by some $\mathrm{Spf}(A)$'s where the A's are special k°-algebras. With the notations of [dJ95, 7.0.1] and [Con99], this means that $\mathfrak{X} \in \mathrm{FS}_{k^\circ}$. We say that $S \subset \mathfrak{X}_\eta$ is subanalytic if for all formal open affine $\mathfrak{U} \subset \mathfrak{X}$, $S \cap \mathfrak{U}_\eta$ is subanalytic in $\mathfrak{U}_\eta$. Then if $\varphi : \mathfrak{X} \to \mathfrak{Y}$ is a morphism between two Noehterian formal scheme of FS_{k°, and S is a subanalytic set of $\mathfrak{X}_\eta$, then $\varphi_\eta(S)$ is a subanalytic set of $\mathfrak{Y}_\eta$. In addition, Theorem 1.2 holds for Noetherian formal schemes $\mathfrak{X} \in \mathrm{FS}_{k^\circ}$.

2.4. **A geometric approach.**

2.4.1. *Generalized ring of fractions.* Generalized ring of fractions have been introduced in [LR00a, section 2]. In this section, we expose this notion of generalized ring of fractions in the context of k-analytic spaces.

Definition 2.16. Let A be a special k°-algebra, and let $f, g \in A$. We set

$$A\langle f/g\rangle = A\langle X\rangle/(f - Xg).$$
$$A[[f/g]] = A[[\rho]]/(f - \rho g).$$

Definition 2.17. A generalized ring of fractions over A is the datum of a certain morphism of special k°-algebras $\varphi : A \to B$ (we set $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$) and of a subset $\mathrm{Dom}(A) \subset \mathfrak{Y}_\eta$, defined inductively as follows.

- The identity morphism $id : A \to A$ is a generalized ring of fractions and $\mathrm{Dom}(A) = \mathfrak{X}_\eta$.
- Let $\varphi : A \to B$ be a generalized ring of fractions and let $f, g \in B$.
  - Let $\varphi_1 : B \to B\langle f/g\rangle$ be the canonical morphism of special k°-algebras. The composition $\varphi_1 \circ \varphi : A \to B \to B\langle f/g\rangle$ is a generalized ring of fractions and
  $$\mathrm{Dom}(B\langle f/g\rangle) = \{x \in (\varphi_1)_\eta^{-1}\big(\mathrm{Dom}(B)\big) \mid g(x) \neq 0\}.$$
  - Let $\varphi_2 : B \to B[[f/g]]$ be the canonical morphism of special k°-algebras. The composition $\varphi_2 \circ \varphi : A \to B \to B[[f/g]]$ is a generalized ring of fractions and
  $$\mathrm{Dom}(B[[f/g]]) = \{x \in (\varphi_2)_\eta^{-1}\big(\mathrm{Dom}(B)\big) \mid g(x) \neq 0\}.$$

Remark 2.18. Let $f, f \in A$ and $A \to A\langle f/g\rangle$ be the associated generalized ring of fractions. Then $\mathrm{Dom}(A\langle f/g\rangle)$ is the analytic domain $\{g \neq 0\}$ and the associated map $\mathrm{Spf}(A\langle f/g\rangle)_\eta \to \mathrm{Spf}(A)_\eta$ induces an isomorphism of k-analytic spaces between $\mathrm{Dom}(A\langle f/g\rangle)$ and the analytic domain of $\mathrm{Spf}(A)_\eta$ defined by $\{|f| \leq |g| \neq 0\}$. This follows from the simple fact that when $g \neq 0$, in $\mathrm{Spf}(A\langle f/g\rangle)_\eta$, the equation $f - Xg = 0$ implies implies that $|f| \leq |g|$, because the variable X satisfies $|X| \leq 1$.

Likewise, the morphism of k-analytic spaces $\mathrm{Spf}(A[[f/g]])_\eta \to \mathrm{Spf}(A)_\eta$ induces an isomorphism of k-analytic spaces between $\mathrm{Dom}(A[[f/g]])$ and the analytic domain of $\mathrm{Spf}(A)_\eta$ defined by $\{|f| < |g|\}$, because in $\mathrm{Spf}(A[[f/g]])_\eta$, $|\rho| < 1$.

Lemma 2.19. *Let $\varphi : A \to B$ be a generalized ring of fractions, $\mathfrak{X} = Spf(A)$, $\mathfrak{Y} = Spf(B)$. Then*

(1) *Dom(B) is an admissible open subset of* $\mathfrak{Y}_\eta$*, which is the complementary set of some closed analytic subset* $V(I)$ *defined by some ideal* $I \subset B$.
(2) φ_η *identifies Dom(B) with an admissible open set of* $\mathfrak{X}_\eta$.

Proof. The proof is a straightforward induction using the above remark. □

In the rest of the text, we will always assimilate $\mathrm{Dom}(B)$ with an admissible open set of either $\mathfrak{X}_\eta$ or $\mathfrak{Y}_\eta$, accordingly to Lemma 2.19 (2).

Remark 2.20. The definition of a generalized ring of fractions and of its domain is a little imprecise. As we have defined it, the definition of $\mathrm{Dom}(B)$ depends on a sequence

$$A = A_0 \to A_1 \to \cdots \to A_n$$

where each $A_i \to A_{i+1}$ is given by

$$A_{i+1} = A_i\langle f_i/g_i\rangle$$

or

$$A_{i+1} = A_i[[f_i/g_i]]$$

for some $f_i, g_i \in A_i$. However it can be proved that if $\varphi : A \to B$ and $\varphi' : A \to B'$ are generalized rings of fractions such that there exists an isomorphism $\psi : B \to B'$ of A-algebras, then $\mathrm{Dom}(B) = \mathrm{Dom}(B')$. As pointed out in [LR00a, remark 2.3 (i)], one way to see this is to remark that if $\varphi : A \to B$ is a generalized ring of fractions, then $\mathrm{Dom}(B) \subset \mathfrak{X}_\eta$ is the set of points $x \in \mathfrak{X}_\eta$ such that there exists an affinoid domain $U \subset \mathfrak{X}_\eta$ which contains x and such that φ_η induces an isomorphism between U and $(\varphi_\eta)^{-1}(U)$.

2.4.2. *Back to subanalytic sets.* The reason why we have introduced generalized ring of fractions is the following fact, which gives a more geometric description of subanalytic sets.

Proposition 2.21. *Let* $S \subset \mathbb{B}^m \times \mathbb{D}^n$ *be a subanalytic set. Then there exists a finite number of generalized ring of fractions*

$$\varphi_i : k^\circ\langle X_1, \dots, X_m\rangle[[\rho_1, \dots, \rho_n]] \to A_i \textit{ for } i = 1 \dots n$$

and for each i*, a constructible subset* $Z_i \subset (\mathfrak{X}_\mathrm{i})_\eta$ *(where* $\mathfrak{X}_\mathrm{i} = Spf(A_i)$*) such that*

$$S_\eta = \bigcup_{i=1}^{n} (\varphi_i)_\eta(Dom(A_i) \cap Z_i).$$

The proof relies on the following lemma.

Lemma 2.22. *Let* $f : \mathcal{O}^m \times \mathfrak{m}^n \to \mathcal{O}$ *be an* L_{an}^D*-term. Then there exists a finite number of generalized rings of fractions*

$$\varphi_i : k^\circ\langle X_1, \dots, X_m\rangle[[\rho_1, \dots, \rho_n]] \to A_i \textit{ for } i = 1 \dots n$$

and for each i*, some closed analytic set* $V(J_i)$ *of* $(\mathfrak{X}_i)_\eta$*, and some function* $f_i \in A_i$ *such that, if we set* $Z_i = (\varphi_i)_\eta(Dom(A_i) \cap V(J_i))$

(1)
$$\mathbb{B}^m \times \mathbb{D}^n = \bigcup_{i=1\dots n} Z_i.$$

(2) *For each* i,
$$((\varphi_i)_{an}))^*(f_{|Z_i}) = (f_i)_{|Dom(A_i) \cap V(J_i)}.$$

Proof. The proof is a straightforward induction on the (inductive) definition of the L^D_{an}-term f. □

Proof. (of Proposition 2.21) *Step 1.* Using the definition of a subanalytic set and the previous lemma, we can prove that there exists a finite number of generalized ring of fractions

$$\varphi_i : k^\circ\langle X_1, \dots, X_m\rangle[[\rho_1, \dots, \rho_n]] \to A_i \text{ for } i = 1 \dots n$$

and for each i a subset $T_i \subset \mathfrak{X}_{i\eta}$ which is finite Boolean combination of inequalities $\{|f| \le |g|\}$ where $f, g \in A_i$, such that

$$S = \bigcup_{i=1}^{n} (\varphi_i)_\eta(\mathrm{Dom}(A_i) \cap T_i).$$

Step 2. We start by a remark. Let C be a special k°-algebra, $f, g \in C$, and let us consider the set T of $\mathrm{Spf}(C)_\eta$ defined by $\{|f| \le |g|\}$. Let us introduce the generalized ring of fractions $C' = C\langle f/g\rangle$. Then $T = \mathrm{Dom}(C') \cup V(f, g)$.

In the same way, if we consider the set T of $\mathrm{Spf}(C)_\eta$ defined by $\{|f| < |g|\}$, and if we introduce the generalized ring of fractions $C' = C[[f/g]]$, then $T = \mathrm{Dom}(C')$.

With the step 1, these two remarks and an induction conclude the proof. □

3. Proof of Theorem 1.2

The following result is due to B. Conrad, but for the convenience of the reader, we reproduce his proof.

Theorem 3.1. [Con99, Theorem 2.3.1] *Let A be a special k°-algebra which is a domain, and $\mathfrak{X} := Spf(A)$. Then $\mathfrak{X}_\eta$ is an irreducible k-analytic space.*

Proof. Since A is excellent [Val75, Val76], replacing A by its normalization, we can assume that A is a normal domain. It then follows that $\mathfrak{X}_\eta$ is normal according to [Con99, 2.1.3]. Then, according to [dJ95, 7.4.1], the ring A corresponds to the analytic functions $f \in \Gamma(\mathfrak{X}_\eta, \mathcal{O}_{\mathfrak{X}_\eta})$ whose norm is bounded by 1. It then follows that $\mathfrak{X}_\eta$ is connected and normal. Hence according to [Con99, Lemma 2.1.4] $\mathfrak{X}_\eta$ is irreducible . □

Remark 3.2. Vladimir Berkovich has recently proved some cohomological finiteness results [Ber13, Theorem 3.1.1] which generalize [Con99, Theorem 2.3.1].

Definition 3.3. Let A be a special k°-algebra, and let $\mathfrak{X} = \mathrm{Spf}(A)$.

(1) Let $I \subset A$ be some ideal. We set

$$V(I) = \{x \in \mathfrak{X}_\eta \mid f(x) = 0 \ \forall f \in I\}.$$

$V(I)$ is a closed analytic set of $\mathfrak{X}_\eta$.

(2) We say that a set $Z \subset \mathfrak{X}_\eta$ is constructible if it is a finite Boolean combination of some $V(I)$'s.

Lemma 3.4. *Let A be a special k°-algebra, let $\mathfrak{X} = Spf(A)$ and let $Z \subset \mathfrak{X}_\eta$ be some constructible subset of $\mathfrak{X}_\eta$. Then Z has finitely many connected components.*

Proof. Step 1. By definition, Z is a finite union of sets of the form $V(\mathfrak{J}_1) \setminus V(\mathfrak{J}_2)$ where the $\mathfrak{J}_i$'s are ideals of A. So we can assume that $Z = V(\mathfrak{J}_1) \setminus V(\mathfrak{J}_2)$ where $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are ideals of A.

Step 2. Replacing A by $A/\mathfrak{J}_1$, we can assume that $Z = \mathfrak{X}_\eta \setminus V(\mathfrak{J})$ where $\mathfrak{J}$ is an ideal of A.

Step 3. Working separately on the irreducible components of $\mathrm{Spec}(A)$ we can assume that A is a domain.

Step 4. We are then reduced to the following situation: A is a special k°-algebra which is a domain, and $Z = \mathfrak{X}_\eta \setminus V(\mathfrak{J})$ for some ideal $\mathfrak{J}$. But in this situation, it follows from the above mentioned result of B. Conrad that $\mathfrak{X}_\eta$ is an irreducible k-analytic space. So according to [Ber90, Corollary 3.3.20], Z is connected. □

Proof of theorem 1.2

Proof. According to Proposition 2.21, we can find some decomposition

$$S = \bigcup_{i=1}^{n} (\varphi_i)_\eta \big(\mathrm{Dom}(A_i) \cap Z_i\big)$$

with $\varphi_i : A \to A_i$ some generalized ring of fractions of A, and the Z_i's are constructible sets of $(\mathfrak{X}_i)_\eta$ where $\mathfrak{X}_i = \mathrm{Spf}(A_i)$. So according to Lemma 2.19 (1), $\mathrm{Dom}(A_i) \cap Z_i$ is a constructible set of $(\mathfrak{X}_i)_\eta$. The result then follows from Lemma 3.4, and from the simple fact that the image of a path-connected space by a continuous map is path-connected. □

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