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Constructibilité dans les espaces de Berkovich

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Résumé

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Dans cette thèse, on s'intéresse à des problèmes de constructibilité en géométrie analytique non archimédienne sur un corps non archimédien k. On étudie certaines parties (semi-analytiques, sous-analytiques...) du point de vue des espaces k-analytiques alors qu'elles n'étaient jusqu'à présent considérées qu'au niveau des points rigides.

On étudie notamment les parties sous-analytiques (et sous-analytiques surconvergentes) en utilisant des points non rigides fournis par les espaces de Berkovich. Cela nous permet d'obtenir de nouvelles preuves de résultats antérieurs, d'établir de nouvelles propriétés et de clarifier une erreur concernant le comportement local des parties sous-analytiques surconvergentes qui n'avait jusque là pas été relevée.

On donne également des théorèmes de finitude pour la cohomologie à support compact de germes $H^q_c((\mathcal{X}^{\mathrm{an}}, S), \mathbb{Q}_l)$ où S est une partie semi-algébrique localement fermée de l'analytifiée d'une k-variété algébrique $\mathcal{X}$. Enfin, on généralise des résultats concernant des applications de tropicalisation d'espaces k-analytiques compacts.

Mots-clefs

constructibilité, non archimédien, espaces de Berkovich, géométrie rigide, semi-analytique, sous-analytique, semi-algébrique, cohomologie étale.

Constructibility in Berkovich spaces

Abstract

In this thesis, we study constructibility problems in non-Archimedean analytic geometry over a non-Archimedean field k. We study some subsets (semianalytic, subanalytic...) in the framework of k-analytic spaces, whereas until now they had only been consider as subsets of rigid k-spaces.

We especially study subanalytic (and overconvergent subanalytic) sets using non-rigid points of Berkovich spaces. With this, we give new proofs of prior results, establish some new properties and clarify a mistake concerning the local behaviour of overconvergent subanalytic sets which had not been noticed until now.

We also give finiteness results for compactly supported cohomology of germs $H_c^q((\mathcal{X}^{\mathrm{an}}, S), \mathbb{Q}_l)$ where S is a locally closed semi-algebraic subset of the analytification of some algebraic k-variety $\mathcal{X}$. Finally, we generalize some results about tropicalization maps of compact k-analytic spaces.

Keywords

constructibility, non-Archimedean, Berkovich spaces, rigid geometry, semianalytic, subanalytic, semialgebraic, etale cohomology.

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Introduction

Sur un corps non archimédien k tel que $\mathbb{Q}_p, \mathbb{C}_p, \mathbb{F}_p((t)), \mathbb{C}((t))$, il est naturel de chercher à étudier des objets de nature analytique.

Motivations générales

Sur un corps algébriquement clos K une variété algébrique X a des groupes de cohomologie étale $H^q(X, \mathbb{Q}_\ell)$ et $H^q_c(X, \mathbb{Q}_\ell)$ qui sont de dimension finie si ℓ est premier à la caractéristique de K. Plus généralement, la stabilité par les six foncteurs de la catégorie des faisceaux constructibles implique que si $f : X \to Y$ est un morphisme de variétés algébriques, la famille $H^q_c(X_y, \mathbb{Q}_\ell)$ se recolle en un faisceau $R^q f_! \mathbb{Q}_\ell$ sur Y qui est localement constant de type fini le long d'un nombre fini de strates constructibles sur Y.

V. Berkovich [Ber93] puis R. Huber [Hub96] ont défini des théories de cohomologie étale pour les variétés analytiques sur k, il est donc tentant de chercher une théorie analogue de faisceaux constructibles pour les variétés analytiques sur k.

Dans cette direction, signalons deux résultats. Si X est un espace k-analytique compact, les groupes $H^q(X, \mathbb{Q}_\ell) \simeq H^q_c(X, \mathbb{Q}_\ell)$ sont de dimension finie. Ce résultat est l'aboutissement de la série d'articles [Ber94, Ber96a, Ber13]. Dans le langage des espaces adiques, si $f : X \to Y$ est lisse et quasi-compact, on a un théorème de stabilité par $R^q f_!$ pour une certaine classe de faisceaux constructibles [Hub96], et dans le cas où $\dim(Y) \leq 1$, et $f : X \to Y$ est seulement quasi-compact, on a des résultats de stabilité par $R^q f_*$ (en caractéristique 0) et $R^q f_!$ pour d'autres classes de faisceaux [Hub98a, Hub98b, Hub07].

Ainsi, pour l'instant, il n'existe pas de classe de faisceaux constructibles stable par $Rf_!$ et Rf_* sans hypothèse forte sur f (lisse ou $\dim(Y) \leq 1$). Si ce travail ne contient pas de tels résultats généraux de stabilité par $Rf_!$ ou Rf_*, nous espérons toutefois qu'il convaincra son lecteur que la bonne définition de faisceau constructible pourrait être celle d'un faisceau localement constant le long de strates sous-analytiques *au sens de L. Lipshitz [Lip93, LR00d].*

Analogies et différences avec $\mathbb{R}$ et $\mathbb{C}$

Signalons les analogies et différences qui existent avec le cas des variétés sur $\mathbb{C}$.

D'une part, d'après les théorèmes de comparaison [Ber93, 7.1.4 et 7.5.3] entre cohomologies étales algébrique et analytique, si $X = \mathcal{X}^{an}$ est l'analytifié d'une variété algébrique sur k, la classe des faisceaux constructibles sur $\mathcal{X}$ produit par le foncteur d'analytification $F \to F^{an}$ une classe stable par les six opérations si l'on se restreint aux analytifiés de morphismes entre variétés algébriques, ce phénomène étant totalement analogue au cas complexe.

Une différence importante est qu'en géométrie complexe, les morphismes propres et compacts coïncident, et de fait les théorèmes de constructibilité en géométrie analytique complexe ou réelle ne marchent que pour des morphismes propres (ce qui équivaut donc à compact). On voit mal de toute manière comment des théorèmes beaucoup plus généraux pourraient exister. Par exemple

$$\begin{array}{ccc} \{z \in \mathbb{C} \mid |z| < 1\} & \xrightarrow{f} & \{z \in \mathbb{C} \mid |z| < 1\} \\ z & \mapsto & e^{\frac{1}{z-1}} \end{array} \tag{1}$$

est un endomorphisme analytique de la boule ouverte, mais ses fibres sont infinies discrètes. En géométrie analytique non archimédienne, il y a une vraie différence entre les morphismes propres et les morphismes compacts. Un morphisme entre espaces k-analytiques est dit propre s'il est topologiquement propre et sans bord [Ber93, 1.5.3(iii)]. De fait il existe beaucoup de morphismes entre espaces k-analytiques topologiquement propres qui ont un bord non vide. C'est pour cette classe de morphismes topologiquement propres qu'on peut espérer des résultats de constructibilité. Par exemple, les briques de base des espaces k-analytiques sont les espaces k-affinoïdes, qui sont compacts (sans être propres), et cela éclaire les résultats de finitude de [Ber13] concernant la cohomologie des affinoïdes. C'est une différence essentielle avec le cas complexes (et réel) où la brique de base est la boule ouverte de rayon 1 qui n'est pas compacte. Pour dire les choses autrement, contrairement au cas archimédien, les briques de base en géométrie non archimédienne sont compactes et pour des raisons tant esthétiques que techniques, il serait satisfaisant d'avoir une théorie de constructibilité qui englobe ces briques de base.

L'idée d'une théorie de constructibilité pour les espaces k-analytiques s'inspire clairement de théories déjà existantes. La théorie des faisceaux constructibles en cohomologie étale ℓ-adique des schémas en est sûrement l'exemple le plus abouti [Del77]. Signalons également une théorie de faisceaux constructibles pour les ensembles semi-algébriques réels (plus généralement pour les structures o-minimales) [Sch03, Chapitre2], et en géométrie sous-analytique réelle [KS94]. Dans toutes ces théories, un faisceau F sur X est constructible s'il existe une *bonne* stratification $\{S_i\}$ de X telle que pour tout i, $F_{|S_i}$ soit localement constant.

Ensembles constructibles, semi-analytiques, sous-analytiques. État de l'art.

En théorie des schémas, on a une bonne définition d'ensemble constructible, et la définition d'un faisceau constructible ne pose pas de problème : une stratification $\{S_i\}$ est *bonne* si chacun des S_i est constructible. En géométrie semi-algébrique, il faut remplacer constructible par semi-algébrique pour définir un faisceau constructible. Remarquons qu'une condition nécessaire pour qu'une telle construction marche est que les ensembles constructibles (resp. semi-algébriques) soient stables par image directe, ce qu'assure le théorème de Chevalley (resp. de Tarski-Seidenberg).

Si l'on cherche pour les espaces k-analytiques une définition analogue il nous faudra donc isoler une classe d'ensemble qui soit stable par image directe. Dans un cadre analytique, trouver une classe *raisonnable* de sous-ensembles stables par images directes n'est pas aisé.

L'exemple (1) illustre que dans le cas complexe, la classe des ouverts bornés de $\mathbb{C}^n$ ne saurait rentrer dans une théorie satisfaisante de constructibilité puisque f est un endomorphisme analytique de la boule ouverte de rayon 1 qui a des fibres infinies discrètes. Le fait que les espaces k-affinoïdes, qui sont les briques de base de la géométrie non-archimédienne, soient des objets de nature compacte rend la situation différente pour les espaces k-analytiques.

Premières approches

Fixons un espace k-affinoïde X et donnons des définitions possibles d'ensemble constructible de X qui s'avèrent ne pas convenir, en expliquant pourquoi.

1. Dire que $S \subset X$ est constructible si c'est une combinaison booléenne de fermés de Zariski de X ? Cette tentative est naturelle par analogie avec le cas des schémas. Même dans le cas le plus simple de la boule unité fermée $X = \mathbb{B}$, si $0 < r < 1$, alors S, la boule fermée de rayon r est obtenue comme l'image de l'inclusion de S dans $\mathbb{B}$, mais S n'est pas une combinaison booléenne de fermés de Zariski de $\mathbb{B}$ (car ce sont les ensembles finis et cofinis).
2. Essayons alors de dire que $S \subset X$ est constructible s'il existe un recouvrement fini $\{X_i\}$ de X par des domaines affinoïdes tel que pour tout i, $S \cap X_i$ soit une combinaison booléenne de fermés de Zariski de X_i. Cela ne marche toujours pas : si on considère
$$\begin{array}{cccc} f: & \mathbb{B}^2 & \to & \mathbb{B}^2 \\ & (x,y) & \mapsto & (x,xy) \end{array}$$
alors $\mathrm{im}(f) = \{|y| \leq |x|\}$. Il est facile de voir que localement autour de l'origine dans $\mathbb{B}^2$, cet ensemble n'est pas une combinaison booléenne de fermés de Zariski (signalons que tout domaine affinoïde contenant l'origine en est un voisinage au sens de la topologie de Berkovich).
3. L'exemple précédent suggère de considérer les combinaisons booléennes d'ensembles du type
$$\{x \in X \mid |f(x)| \leq |g(x)|\}$$
où $f, g \in \mathcal{A}$. Appelons ces ensembles **ensembles semi-analytiques** [1]. Cette tentative est naturelle, les ensembles semi-analytiques étant l'analogue des ensembles semi-algébriques réels, mais ne marche toujours pas. Considérons $g = \sum_{n \in \mathbb{N}} g_n T^n$ une série à coefficients dans k de rayon de convergence exactement r avec $0 < r < 1$ et telle que $\|g\| < 1$. Si
$$\begin{array}{cccc} \varphi: & \mathbb{B}_r & \to & \mathbb{B}^2 \\ & x & \mapsto & (x, g(x)) \end{array}$$
alors $\mathrm{im}(\varphi)$ n'est pas semi-analytique dans $\mathbb{B}^2$ (cf. par exemple 1.2.6).
4. Qu'à cela ne tienne, on pourrait considérer les $S \subset X$ pour lesquels il existe un recouvrement fini $\{X_i\}$ de X par des espaces affinoïdes tel que $S \cap X_i$ soit semi-analytique pour tout i. Un ensemble vérifiant cette propriété sera dit **rigide semi-analytique** . Cela ne marche toujours pas : il existe des morphismes entre espaces affinoïdes dont l'image n'est pas rigide semi-analytique (voir par exemple 1.2.10).

Remarquons la chose suivante. Si $f, g \in \mathcal{A}$, considérons

$$Y := \{(x,t) \in X \times \mathbb{B} \mid f(x) = tg(x)\}$$

1. Dans la littérature [Lip93, Sch94a], ces ensembles sont appelés *globalement semi-analytiques*. Et le terme semi-analytique est utilisé pour un classe plus grande, mais qui n'est pas la même dans [Sch94a, 1.3.7.1] et [Lip93, 4.5]. Nous justifions notre terminologie par la raison suivante. Les classe des parties définies par des inégalités $|f| \leq |g|$ où f et g sont des fonctions analytiques nous semble la classe la plus simple, et la plus facile à appréhender de celles que nous étudierons. Lui donner le nom le plus simple nous a semblé naturel.

et $\varphi : Y \to X$ la projection suivant la première coordonnée. Alors $\mathrm{im}(\varphi) = \{x \in X \mid |f(x)| \leq |g(x)|\}$, de sorte qu'une classe d'ensembles stable par image entre morphismes affinoïdes devra nécessairement contenir la classe des ensembles semi-analytiques.

Élimination des quantificateurs

Dans le cadre algébrique, la situation est plus facile. Soit $\mathcal{X} = \mathrm{Spec}(A)$ un k-schéma affine de type fini. On dira que $S \subset \mathcal{X}^{an}$ est **semi-algébrique** si c'est une combinaison booléenne d'ensembles

$$\{x \in \mathcal{X}^{an} \mid |f(x)| \leq |g(x)|\}$$

où $f, g \in A$. Le résultat suivant est prouvé dans [Duc03] et découle du théorème d'élimination des quantificateurs pour ACVF. Nous détaillerons ce résultat dans la section 0.4

Théorème. *Soit $\mathcal{X}, \mathcal{Y}$ deux k-schémas affines de type fini et $\varphi : \mathcal{X} \to \mathcal{Y}$ un morphisme de k-schémas. Si S est un ensemble semi-algébrique de $\mathcal{X}^{an}$, $\varphi^{an}(S)$ est un ensemble semi-algébrique de $\mathcal{Y}^{an}$.*

Ensembles sous-analytiques [2] selon Schoutens et Lipshitz

À la Schoutens

On suppose que k est algébriquement clos [3]. On note $k\langle\langle X_1, \ldots, X_n\rangle\rangle$ la k-algèbre des séries entières surconvergentes sur $\mathbb{B}^n$, i.e. dont le rayon de convergence est $r > 1$. Fixons $\mathcal{A}$ une algèbre k-affinoïde et X l'espace rigide associé. On considère l'opérateur D :

$$D : \begin{array}{ccc} k \times k & \to & k \\ (x, y) & \mapsto & \begin{cases} \frac{x}{y} \text{ si } |x| \leq |y| \neq 0 \\ 0 \text{ sinon.} \end{cases} \end{array}$$

H. Schoutens définit dans [Sch94a] $\mathcal{A}\langle\langle D\rangle\rangle$ comme la plus petite k-algèbre de fonctions $f : X \to k$ telle que

- $\mathcal{A}\langle\langle D\rangle\rangle$ contient les fonctions induites par $\mathcal{A}$.
- Si $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$ alors $D(f, g) \in \mathcal{A}\langle\langle D\rangle\rangle$.
- Si on se donne $f_1, \ldots, f_n \in \mathcal{A}\langle\langle D\rangle\rangle$ vérifiant $\|f_i\|_{\sup} \leq 1$ et si $F \in k\langle\langle X_1, \ldots, X_n\rangle\rangle$ alors $F(f_1, \ldots, f_n) \in \mathcal{A}\langle\langle D\rangle\rangle$.

Théorème ([Sch94a]). *Si $S \subset X$ on a équivalence entre*

- *S est une combinaison booléenne d'ensembles définis par des inégalités $|f| \leq |g|$ pour $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$.*
- *Il existe une partie semi-analytique $T \subset X \times \mathbb{B}^n$ définie par des inégalités $|f| \leq |g|$ qui sont surconvergentes par rapport aux variables de $\mathbb{B}^n$, tel que $S = \pi(T)$ où $\pi : X \times \mathbb{B}^n \to X$ est la première projection.*

Nous dirons que S est **sous-analytique surconvergent** [4] s'il vérifie l'une de ces conditions.

2. Signalons qu'il existe une théorie des parties semi-analytiques et sous-analytiques des variétés analytiques réelles (voir [BM88] pour une belle introduction). Cette théorie est antérieure au cas non archimédien, et tant au niveau des définitions que des résultats, les travaux que nous allons présenter s'inspirent du cas réel.

3. Cette hypothèse n'est en rien indispensable mais permet de simplifier l'exposition.

4. Là encore, nous avons choisi de ne pas utilisé la terminologie que l'on trouve dans la littérature [Sch94a], où les parties que nous appelons sous-analytiques surconvergentes sont dites globalement

À la Lipshitz

La construction de Lipshitz [Lip93, LR00d] repose sur l'introduction d'une k-algèbre $S_{m,n}$ de fonctions analytiques

$$\left((k^{\mathrm{alg}})^{\circ}\right)^{m} \times \left((k^{\mathrm{alg}})^{\circ\circ}\right)^{n} \to \ k^{\mathrm{alg}}$$

qui ont de bonnes propriétés. Quand $n = 0$, on retrouve $S_{m,0} = k\{X_1, \ldots, X_m\}$, mais quand $n \geq 1$, la situation est plus délicate, l'algèbre de toutes les fonctions analytiques sur $\mathbb{B}^m \times (\overset{\circ}{\mathbb{B}})^n$ n'étant par exemple pas noethérienne. Nous renvoyons à [LR00d, 2.1.1] et la section 0.5 de ce travail pour plus de détail. Précisons toutefois que

$$k\langle X_1, \ldots, X_m, \rho_1, \ldots, \rho_n\rangle \subset S_{m,n} \subset k\otimes_{k^\circ} k^\circ\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_n]] \subset \Gamma\big(\mathbb{B}^m \times (\overset{\circ}{\mathbb{B}})^n, \mathcal{O}_{\mathbb{B}^m\times(\overset{\circ}{\mathbb{B}})^n}\big),$$

ces inclusions étant strictes dès que $n \geq 1$ et k n'est pas de valuation discrète. Signalons aussi que $k \otimes_{k^\circ} k^\circ\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_n]]$ n'est pas noethérien quand k n'est pas de valuation discrète, alors que $S_{m,n}$ l'est (c'est l'un des intérêts de ces algèbres). Afin de donner une idée précise de ces algèbres, mentionnons que si k est de valuation discrète,

$$S_{m,n} = k \otimes k^\circ\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_n]]$$

et que si $k = \mathbb{C}_p$ par exemple,

$$S_{m,n} = \mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p^{unr}} \mathbb{Q}_p^{unr}\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_n]].$$

Les fonctions de $S_{m,n}$ sont utilisées avec les deux opérateurs

$$D_0 : k^{\mathrm{alg}} \times k^{\mathrm{alg}} \to k^{\mathrm{alg}},\ (x, y) \mapsto \begin{cases} \frac{x}{y} \text{ si } |x| \leq |y| \neq 0 \\ 0 \text{ sinon} \end{cases}$$

$$D_1 : k^{\mathrm{alg}} \times k^{\mathrm{alg}} \to k^{\mathrm{alg}},\ (x, y) \mapsto \begin{cases} \frac{x}{y} \text{ si } |x| < |y| \neq 0 \\ 0 \text{ sinon} \end{cases}$$

pour définir inductivement, l'algèbre des **D-fonctions** par les conditions :
- Les fonctions induites par $\mathcal{A}$ sont des D-fonctions.
- Si f, g sont des D fonctions, alors $D_0(f, g)$ et $D_1(f, g)$ sont des D-fonctions.
- Si $f_1, \ldots, f_m, g_1, \ldots, g_n$ sont des D-fonctions, $F \in \mathcal{A}\langle D\rangle$ et si pour tout $x \in X$, $|f_i(x)| \leq 1$ et $|g_j(x)| < 1$ alors $F(f_1, \ldots, f_m, g_1, \ldots, g_n)$ est une D-fonctions.

On dira qu'un ensemble $S \subset X$ est **sous-analytique** si c'est une combinaison booléenne de parties définies par des inégalités $|f| \leq |g|$ avec f, g des D-fonctions.

Théorème ([Lip93]). *Les ensembles sous-analytiques sont stables par projections. Ce qui peut s'incarner dans les deux énoncés suivants :*

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fortement sous-analytique. Nous avons fait ce choix d'une part pour des raisons esthétiques, notre terminologie nous semblant plus concise et plus imagée. L'autre raison est que Schoutens utilise dans ses articles le terme *fortement sous-analytique* pour désigner différente types de parties : les parties globalement fortement sous-analytiques, fortement sous-analytiques, localement fortement sous-analytiques, globalement fortement D-semi-analytiques, fortement D-semi-analytiques. Cet emploi était justifié par les résultats [Sch94a, théorème p.270, prop. 4.2] selon lesquels ces parties sont toutes équivalentes. Comme nous l'expliquerons au chapitre 1, ces résultats sont faux : les classes mentionnées ci-dessus ne sont pas les mêmes. De sorte que l'ancienne terminologie prêtait à confusion, d'autant plus que Schoutens dans ses articles [Sch94c, Sch94b] utilisait le terme *fortement sous-analytique* indifféremment pour toutes ces classes de parties.

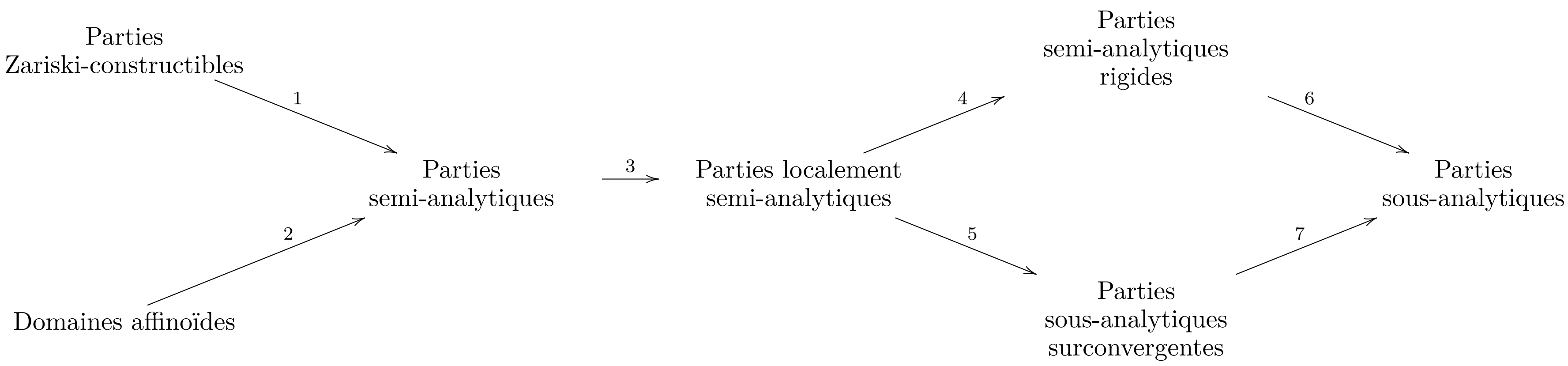
- *Si* $S \subset X \times \mathbb{B}^n$ *est sous analytique et* $\pi : X \times \mathbb{B}^n \to X$ *est la première projection, alors* $\pi(S)$ *est sous-analytique.*
- *Si* $f : X \to Y$ *est un morphisme entre espaces affinoïdes,* $S \subset X$ *un ensemble sous-analytique, alors* $f(S)$ *est un ensemble sous-analytique de* Y*.*

Signalons enfin qu'un ensemble $S \subset X$ qui est sous-analytique surconvergent est forcément sous-analytique. Mais la réciproque est fausse (cf la proposition 1.2.10 pour un résultat encore plus fin).

L'idée des preuves

Les preuves des résultats de Schoutens et Lipshitz marchent suivant un schéma commun. Dans un premier temps, via des théorèmes du type *Préparation de Weierstrass*, on réussit à algébriser les fonctions mises en jeu, et on peut alors utiliser le théorème d'élimination des quantificateurs de ACVF.

A ce stade, nous voulons résumer les différentes classes de parties que nous avons considérées pour un espace k-affinoïde.



Dans ce diagramme une flèche Classe A $\to$ Classe B signifie qu'une partie qui est dans la Classe A est aussi dans la classe B. L'inclusion $\overset{2}{\to}$ découle du théorème de Gerritzen-Grauert. L'inclusion $\overset{3}{\to}$ est prouvée en 1.1.47, et découle du fait que les parties sous-analytiques surconvergentes se recollent bien pour la topologie de Berkovich. Toutes les autres inclusions découlent des définitions. Nous voulons insister sur le fait que **génériquement, toutes les inclusions sont strictes** et qu'il n'y a pas d'inclusion entre les parties semi-analytiques rigides et sous-analytiques surconvergentes(ni dans un sens ni dans l'autre). Signalons les exceptions notables :

– Si $\dim(X) = 0$ toutes les classes coïncident. Dès que $\dim(X) \geq 1$ les inclusions 1 et 2 sont strictes.
– Quand $X = \mathbb{B}$ est la boule unité fermée les inclusions 3–7 sont en fait des égalités (cf la section 0.3.5.
– Dès que X contient un point rationnel x (i.e. à coefficients dans k) régulier tel que $\dim_x(X) \geq 2$, alors 3,4,6 et 7 sont stricts. Pour 3, voir 1.2.6, pour 4 et 7 voir 1.2.4.
– En revanche, si $\dim(X) \leq 2$, X est régulier et k algébriquement clos, 5 est une égalité (théorème 1.3.12).
– Si X contient un point rationnel x (i.e. à coefficients dans k) régulier tel que $\dim_x(X) \geq 3$, toutes les inclusions sont strictes (la seule qui manquait étant 5, voir pour cela 1.2.10).

FIGURE 1 – La Jungle de toutes les parties

Ensembles semi(-sous)-analytiques (surconvergents) vs Espaces de Berkovich. Un panorama de nos résultats.

Nous avons énoncé les résultats de Schoutens et Lipshitz dans le langage de la géométrie rigide, et c'est de fait dans ce langage qu'ils ont été énoncés (ne serait-ce que pour des raisons chronologiques [5]) et développés depuis lors. Ce travail se veut un pont entre les diverses théories d'ensembles semi/sous-analytiques et les espaces k-analytiques. Nous espérons convaincre le lecteur que chacune de ces théories peut apporter à l'autre. En voici quatre résultats représentatifs de cette thèse, pour illustrer cette idée.

Dans le chapitre 1 on obtient une preuve différente, qui nous semble à la fois plus naturelle et instructive du théorème de projection des parties sous-analytiques surconvergentes de Schoutens énoncé page 10, qui utilise la compacité des espaces affinoïdes (vu comme espaces k-analytiques). L'utilisation de points spécifiques aux espaces k-analytiques (un point de type 2 en l'occurrence) nous permet de donner des contre-exemples aux autres résultats de [Sch94a].

Dans le chapitre 2 nous montrons comment les théorèmes de finitude [Ber13] concernant la cohomologie d'un affinoïde peuvent s'étendre. Signalons par exemple

Théorème A. *Supposons k algébriquement clos, $\ell \neq car(\tilde{k})$, $f : X \to Y$ un morphisme d'espace k-affinoïdes et $U \subset Int(X/Y)$ un domaine affinoïde de X. Alors les groupes*

$$H_c^q\Big((Y, f(U)), \mathbb{Z}/\ell\mathbb{Z}\Big)$$

sont finis, ces groupes désignant les groupes de cohomologie du germe $(Y, f(U))$.

En particulier, si $f(U)$ est un fermé de Zariski dans un domaine analytique de Y, ces groupes coïncident avec la *vraie* cohomologie de $f(U)$, vu comme espace k-analytique. Nous obtenons un résultat de finitude analogue pour la cohomologie du germe de parties semi-algébriques qui a été utilisé par E Hrushovski et F. Loeser dans [HL11].

Dans le chapitre 3, on considère la théorie de la dimension des ensembles sous-analytiques, introduite et déjà étudiée dans [LR00a]. En utilisant des points non rigides des espaces de Berkovich, on donne de nouvelles preuves de la plupart des résultats de [LR00a], et on démontre également de nouvelles propriétés, qui nous seront utiles au chapitre 4.

Au chapitre 4, nous introduisons et étudions une théorie de la dimension pour les parties sous-analytiques de $\mathbb{B}^m \times \mathbb{R}_+^n$. Nous obtenons comme corollaire direct le résultat suivant, qui était notre motivation première :

Théorème B. *Soit X un espace k-affinoïde, $f_1, \dots, f_n$, n fonctions analytiques sur X et*

$$\begin{array}{rccc} |f| : & X & \to & \mathbb{R}_+^n \\ & x & \to & (|f_i(x)|)_{i=1\dots n} \end{array}$$

Alors $|f|(X)$ est une union finie de polyèdres [6] de $\mathbb{R}_+^n$, et

$$\dim(|f|(X)) \leq \dim(X).$$

5. L'article de L. Lisphitz [Lip93] date de 1993, seulement trois ans après le premier texte [Ber90] de V. Berkovich traitant des espaces k-analytiques.

6. Un polyèdre de $\mathbb{R}_+^n$ signifiant ici une intersection finie de parties définies par des inégalités $\lambda \prod_{i=1}^n \gamma_i^{a_i} \leq \mu \prod_{i=1}^n \gamma_i^{b_i}$ avec $\lambda, \mu \in \mathbb{R}_+$ et $a_i, b_i \in \mathbb{N}$.

Ce résultat, qui s'énonce purement dans le langage des espaces k-analytiques est démontré en utilisant les ensembles sous-analytiques.

Faisons la remarque importante suivante. Nous aurons trois manières de *décrire* une partie (par exemple semi-analytique, sous-analytique surconvergente, sous-analytique...) du polydisque fermé [7] :

- comme une partie de $\left((k^{\mathrm{alg}})^{\circ}\right)^n$.
- comme une partie du disque vu comme l'espace rigide $\mathrm{Sp}(k\langle X_1,\dots,X_n\rangle)$, ce qui revient à considérer $\left((k^{\mathrm{alg}})^{\circ}\right)^n$ modulo l'action de $\mathrm{Gal}(k^{sep}/k)$.
- comme une partie de l'espace de Berkovich $\mathbb{B}^n = \mathcal{M}(k\langle X_1,\dots,X_n\rangle)$.

Il n'est *a priori* pas clair qu'une propriété prouvée pour les parties semi-analytiques [8] de $\mathbb{B}^n$ soit également valable pour les parties semi-analytiques de $\left((k^{\mathrm{alg}})^{\circ}\right)^n$ (on pense par exemple à la proposition 1.2.4). Nous montrerons dans la section 1.2.4 que c'est bien le cas, et qu'il y a une correspondance entre les parties semi-analytiques vues comme parties de $\left((k^{\mathrm{alg}})^{\circ}\right)^n$ ou comme parties de $\mathbb{B}^n$.

Contenu détaillé de la thèse

Chapitre 1

On considère X un espace k-affinoïde et $\mathcal{A}$ son algèbre affinoïde associée. Nous avons déjà mentionné qu'il y a équivalence entre les parties de la forme $\pi(T)$ où $T \subset X \times \mathbb{B}^n$ est semi-analytique surconvergent par rapport aux variables de $\mathbb{B}^n$, et les parties définies par une combinaison booléenne d'inégalités $|f| \le |g|$ où $f,g \in \mathcal{A}\langle\langle D\rangle\rangle$, $\mathcal{A}\langle\langle D\rangle\rangle$ étant la plus petite algèbre de fonctions contenant $\mathcal{A}$, stable par l'opérateur D (qui correspond à prendre des divisions) et stable par composition avec des fonctions analytiques surconvergentes, la classe ainsi définie étant celle des ensembles sous-analytiques surconvergents. Un inconvénient de la seconde caractérisation est que les fonctions de $\mathcal{A}\langle\langle D\rangle\rangle$ sont difficiles à manipuler. Elles ne sont *a priori* pas continues, que ce soit pour la topologie de Berkovich ou pour la topologie canonique sur l'ensemble des points rigides, à cause de l'opérateur D. Pour contourner cette difficulté, nous utiliserons la démarche suivante : si $f,g \in \mathcal{A}$, on considère

$$Y := \mathcal{M}(\mathcal{A}\{T\}/(f-Tg)) \xrightarrow{\varphi} X = \mathcal{M}(\mathcal{A}). \tag{2}$$

Cela revient en un certains sens à éclater l'idéal (f,g), et φ induit un isomorphisme entre $\{y \in Y \mid g(y) \neq 0\}$ et $\{x \in X \mid |f(x)| \le |g(y)| \neq 0\}$, et sur ces domaines, $D(f,g)$ correspond à la nouvelle fonction T. C'est cette remarque permet de remplacer l'utilisation des fonctions de $\mathcal{A}\langle\langle D\rangle\rangle$ par une construction plus géométrique : nous appelons **donnée constructible** un morphisme $X' \xrightarrow{\varphi} X$ obtenu comme une composée

$$X_n \xrightarrow{\varphi_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\varphi_1} X_0$$

où chaque φ_i est un morphisme du type (2) et $X_0 = X$ (en pratique la définition est plus complexe, puisqu'il faut tenir compte du phénomène de surconvergence intrinsèque à $\mathcal{A}\langle\langle D\rangle\rangle$). On dit alors qu'un ensemble $S \subset X$ est **constructible surconvergent** s'il s'écrit $\varphi(S)$ où $\varphi : X' \to X$ est une donnée constructible, et S est un ensemble semi-analytique *convenable* de X'.

Theorème 1. *Une partie $S \subset X$ est sous-analytique surconvergent si et seulement si S est constructible surconvergent.*

7. Plus généralement d'un espace affinoïde.
8. Ou sous-analytiques surconvergents, localement semi-analytique, sous-analytique...

Ce théorème n'est autre que l'équivalence (1) $\Leftrightarrow$ (3) de [Sch94a, Quantifier élimination theorem p.270], modulo le fait (prouvé en 1.1.7) que les constructibles surconvergents et les combinaisons booléennes d'inégalités $|f| \leq |g|$ entre fonctions de $\mathcal{A}\langle\langle D\rangle\rangle$ coïncident. Cependant la preuve que nous donnons utilise de manière essentielle la compacité des espaces k-affinoïdes et est en ce sens nouvelle. Pour ce faire, nous démontrons en 1.1.3 des résultats de type préparation de Weierstrass au dessus d'un anneau de Banach et pour un rayon arbitraire. En 1.1.6 nous donnons des résultats analogues en autorisant des inégalités $|f| \leq \lambda|g|$, avec $\lambda \in \mathbb{R}_+$ qui sont valables même quand le corps est trivialement valué.

Dans la partie 1.2, nous donnons des contre-exemples aux autres résultats de [Sch94a] concernant les propriétés locales des ensembles sous-analytiques surconvergents. En particulier la proposition [Sch94a, 4.2] stipule que si $\{X_i\}$ est un recouvrement affinoïde fini de X, et $S \subset X$, tel que $S \cap X_i$ est sous-analytique surconvergent dans X_i pour tout i, alors S est sous-analytique surconvergent dans X. Nous montrons dans la partie 1.2.2 que ça n'est pas le cas. En fait nous prouvons même un résultat plus fort (proposition 1.2.4) à savoir qu'il existe des ensembles semi-analytiques rigides qui ne sont pas sous-analytiques surconvergents. Le contre-exemple que l'on présente utilise un point de $\mathbb{B}^2$ qui n'est pas rigide (c'est le point de Gauss d'une courbe plongée dans $\mathbb{B}^2$ non prolongeable) ainsi que les bénéfices géométriques de la caractérisation des ensembles sous-analytiques surconvergents en terme de constructibles surconvergents. Ainsi les ensembles sous-analytiques surconvergents ne peuvent pas se recoller pour la topologie des recouvrement affinoïdes finis, mais nous expliquons que c'est la topologie de Berkovich qu'il faut considérer

Proposition 2. *Soit $S \subset X$, alors S est sous-analytique surconvergent si et seulement si pour tout $x \in X$ il existe un voisinage affinoïde U de x tel que $S \cap U$ est sous-analytique surconvergent dans U.*

Cette propriété nous permet d'étendre la définition des ensembles sous-analytiques surconvergents aux bons espaces k-analytiques (plus nécessairement affinoïdes) et nous étudions des propriétés globales en 1.1.5.

Dans [Sch94a] sont énoncées d'autres caractérisations des ensembles sous-analytiques surconvergentes dont on montre par des méthodes similaires qu'elles sont fausses. Tout cela est clarifié par la figure 1.1 page 82.

Les résultats erronés de [Sch94a] furent utilisés ensuite dans [Sch94c] et [Sch94b], de sorte que les preuves et même certains des énoncés de ces articles sont faux. En 1.3 nous corrigeons ainsi les énoncés et résultats de [Sch94b]

Theorème 3. *Soit X un bon espace k-analytique quasi-lisse de dimension ≤ 2 avec k algébriquement clos, et $S \subseteq X$. Alors S est sous-analytique surconvergent si et seulement si S est localement (pour la topologie de Berkovich) semi-analytique.*

Chapitre 2

On fixe un nombre premier $\ell \neq \mathrm{car}(\tilde{k})$.

Dans un premier temps, supposons k algébriquement clos. Si X est un espace k-analytique, $S \subset X$ et F un faisceau étale sur X, Berkovich a défini des groupes de cohomologie pour le germe (X, S)

$$H^q_c((X,S), F_{(X,S)}).$$

Si S est un fermé de Zariski dans un domaine analytique de X, Berkovich a prouvé que ces groupes coïncident avec les *vrais* groupes de cohomologie $H^q_c(S, F_{|S})$. Nous prouvons (proposition 2.2.3) que si X est un espace k-affinoide, et $S \subset X$ un ensemble semi-analytique localement fermé, les groupes $H_c^q((X,S),\mathbb{Z}/\ell^n\mathbb{Z})$ sont finis, puis ce résultat de finitude est étendu aux cas suivants :

1. X est un espace k-analytique compact et S est semi-analytique rigide (proposition 2.3.1).
2. $X=\mathcal{X}^{\mathrm{an}}$ où $\mathcal{X}$ est un k-schéma séparé de type fini et S une partie semi-algébrique localement fermée de $\mathcal{X}^{\mathrm{an}}$, qui est en fait un cas particulier de :
3. $X=\mathcal{X}^{\mathrm{an}}$ où $\mathcal{A}$ est une algèbre k-affinoïde, $\mathcal{X}$ est un $\mathcal{A}$-schéma séparé de type fini, et S est un sous-ensemble semi-algébrique localement fermé de $\mathcal{X}^{\mathrm{an}}$ (proposition 2.3.2).
4. X est un espace k-analytique compact et S une partie sous-analytique surconvergente localement fermée de X (proposition 2.4.1).

Dans la section 2.6, on explique comment passer des coefficients de torsions $\mathbb{Z}/\ell^n\mathbb{Z}$ aux groupes $H_c^q((X,S),\mathbb{Q}_\ell)$. Nous avons choisi d'en donner une preuve le plus simple possible, faute de référence disponible dans la littérature. Tous ces résultats se résument ainsi

Theorème 4. *Soit $\mathcal{A}$ une algèbre k-affinoïde (k n'est pas forcément algébriquement clos), $\mathcal{X}$ un $\mathcal{A}$-schéma séparé de type fini de dimension d, U une partie semi-algébrique localement fermée de $\mathcal{X}^{an}$. On note $\overline{\mathcal{X}^{an}}:=\mathcal{X}^{an}\widehat{\otimes}_k\widehat{k^{alg}}$ et on pose $\pi:\overline{\mathcal{X}^{an}}\to\mathcal{X}^{an}$ et $\overline{U}=\pi^{-1}(U)$.*

1. *Les groupes $H_c^i(\overline{U},\mathbb{Q}_\ell)$ sont des $\mathbb{Q}_\ell$-espaces vectoriels de dimension finis, munis d'une Gal(k^{sep}/k)-action, et $H_c^i(\overline{U},\mathbb{Q}_\ell)=0$ pour $i>2d$.*
2. *Soit $V\subset U$ une partie semi-algébrique ouverte dans U, et soit $F=U\setminus V$. Il y a alors une suite exacte longue Gal(k^{sep}/k)-équivariante*

$$\cdots\to H_c^i(\overline{V},\mathbb{Q}_\ell)\longrightarrow H_c^i(\overline{U},\mathbb{Q}_\ell)\longrightarrow H_c^i(\overline{F},\mathbb{Q}_\ell)\longrightarrow H_c^{i+1}(\overline{V},\mathbb{Q}_\ell)\to\cdots$$

3. *Pour tout entier n il y a des isomorphismes canoniques Gal(k^{sep}/k)-équivariants :*

$$\bigoplus_{i+j=n} H_c^i\left(\overline{U},\mathbb{Q}_\ell\right)\otimes_{\mathbb{Q}_\ell} H_c^j\left(\overline{V},\mathbb{Q}_\ell\right)\simeq H_c^n\left(\overline{U\times V},\mathbb{Q}_\ell\right)$$

Ce problème nous a été posé par F. Loeser et a été utilisé dans [HL11] pour l'étude de la fibre de Milnor associée à un morphisme $f:X\to\mathbb{A}^1_{\mathbb{C}}$ où X est une variété complexe algébrique. Le corps utilisé est $k=\mathbb{C}((t))$. Notons que ces résultats sont également valables si U est une partie sous-analytique surconvergente localement fermée d'un bon espace k-analytique d'après les résultats de la section 2.4.

Enfin dans la section 2.5, on explique comment les résultats de finitude pour $H_c^q((X,S),\mathbb{Z}/\ell^n\mathbb{Z})$ ont des analogues en cohomologie étale des espaces adiques.

Chapitre 3

Ce chapitre est consacré à la dimension des ensembles sous-analytiques de $\mathbb{B}^n$. Dans [LR00a] la dimension d'un ensemble sous-analytique S de $\left((k^{\mathrm{alg}})^\circ\right)^n$ est définie comme le plus petit entier d tel qu'il existe une projection selon des coordonnées $\pi:\left((k^{\mathrm{alg}})^\circ\right)^n\to\left((k^{\mathrm{alg}})^\circ\right)^d$ telle que $\pi(S)$ soit d'intérieur non-vide. Comme nous l'avons dit plus haut, à une partie sous-analytique $S\subset\left((k^{\mathrm{alg}})^\circ\right)^n)$ correspond une partie sous-analytique de l'espace de Berkovich $\mathbb{B}^n$ que nous noterons S_{Berko}. Nous donnons dans le théorème 3.1.14 huit caractérisations équivalentes de cette dimension, certaines déjà présentes dans [LR00a], d'autres nouvelles, citons en particulier

Proposition 5.

$$\dim(S) = \max_{x \in S_{Berko}} d(\mathcal{H}(x)/k). \tag{3}$$

Où si $k \to K$ est une extension non archimédienne, $d(K/k) = \text{tr.deg.}(\tilde{K}/k) + \dim_{\mathbb{Q}}(\sqrt{|K^*|}/\sqrt{|k^*|})$. La preuve de (3) utilise de manière essentielle les points x de $\mathbb{B}^n$ dits d'Abhyankar qui réalisent le maximum $d(\mathcal{H}(x)/k) = n$. C'est sur l'égalité 3 que nous nous reposerons principalement. Remarquons que cette égalité est naturelle dans le sens où cette même formule donne la dimension des espaces k-analytiques [Ber93, 2.5.5]. Nous obtenons par exemple une preuve sensiblement différente de celle de [LR00a] de l'inégalité

$$\dim(\overline{S} \setminus S) < \dim(S)$$

pour S une partie sous-analytique non-vide. Nous obtenons également des résultats nouveaux et qui nous serviront au chapitre 4 :

1. Soit $S \subset \left((k^{\text{alg}})^\circ\right)^{n+m}$ une partie sous-analytique et $\pi : S \to \left((k^{\text{alg}})^\circ\right)^n$ la première projection, définie par $(x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (x_1, \ldots, x_n)$. Si toutes les fibres non-vides de π ont la même dimension d, alors

$$\dim(S) = \dim(\pi(S)) + d.$$

2. Si $f : S \to T$ est une application sous-analytique (i.e. dont le graphe est sous-analytique), alors $\dim(f(S)) \leq \dim(S)$. En particulier, si S et T sont en bijection par une application sous-analytique, $\dim(S) = \dim(T)$.
3. $\dim(S \times T) = \dim(S) + \dim(T)$.

Ces résultats sont fortement inspirés dans leur formulation et dans leur preuve de l'article [vdD89] qui donne des résultats analogues pour la dimension des ensembles définissables dans la théorie des corps valués algébriquement clos, que nous présentons dans la section 3.2.

Chapitre4

Le but de ce chapitre était initialement de montrer le théorème B page 14 en utilisant le théorème d'élimination des quantificateurs de Lipshitz pour les ensembles sous-analytiques, et en particulier le fait que l'on peut également éliminer les variables du groupe des valeurs $\Gamma = \sqrt{|k^\times|} \subset \mathbb{R}_+^*$. Nous montrons ce théorème avec une approche plus générale. On définit d'abord les ensembles sous-analytiques de $\left((k^{\text{alg}})^\circ\right)^m \times \Gamma^n$. Pour $n = 0$, on retrouve notre définition précédente d'ensemble sous-analytique de $\left((k^{\text{alg}})^\circ\right)^m$, et pour $m = 0$, les ensembles sous-analytiques sont juste les parties définissables pour la structure o-minimale de groupe abélien totalement ordonné de Γ ; ce sont donc les combinaisons booléennes définies par les inégalités de la forme $\sum_{i=1}^n a_i\gamma_i \leq \sum_{i=1}^n b_i\gamma_i$, avec $a_i, b_i \in \mathbb{N}$.

Nous définissons pour une partie sous-analytique $S \subset \left((k^{\text{alg}})^\circ\right)^m \times \Gamma^n$ une dimension $\dim_{\mathbb{N}}(S) \in \mathbb{N} \cup -\infty$ qui se comporte bien

Theorème 6. *1. Si $S \subset \left((k^{alg})^\circ\right)^m$, $\dim_{\mathbb{N}}(S)$ correspond à la dimension définie au chapitre 3.*

2. Si $S \subset (\Gamma \cup \{0\})^n$, $\dim_{\mathbb{N}}(S)$ est la dimension classique *telle que définie pour les structures o-minimales [vdD98].*

3. Si $f : S \to T$ est une application sous-analytique, $\dim_{\mathbb{N}}(f(S)) \leq \dim_{\mathbb{N}}(S)$. En particulier $\dim_{\mathbb{N}}$ est invariant par bijection sous-analytique.

4. $\dim_{\mathbb{N}}(S \times T) = \dim_{\mathbb{N}}(S) + \dim_{\mathbb{N}}(T)$.

Le théorème B est alors obtenu comme un corollaire (voir 4.3.1) de 3. et du théorème d'élimination des quantificateurs pour les parties sous-analytiques de $\left((k^{\mathrm{alg}})^{\circ}\right)^m \times \Gamma^n$.

En fait, la construction et les propriétés de l'invariant $\dim_{\mathbb{N}}$ sont obtenus grâce à l'étude d'un autre invariant plus fin. Si S est une partie sous-analytique de $(K^{alg})^m \times \Gamma^n$, on lui associe une **dimension mixte** $\dim(S) \subset \mathbb{N}^2$, qui est une partie finie de $\mathbb{N}^2$, stable par $\leq$ (par exemple $(2,1) \leq (3,5)$ donc si $(3,5) \in \dim(S)$, on a aussi $(2,1) \in \dim(S)$). De manière précise, $(d_1, d_2) \in \dim(S)$ signifie que S contient une sous-partie définissablement isomorphe à un ouvert de $((k^{\mathrm{alg}})^{\circ})^{d_1} \times \Gamma^{d_2}$. Nous établissons un théorème de décomposition cellulaire pour les ensembles sous-analytiques de $\left((k^{\mathrm{alg}})^{\circ}\right)^m \times \Gamma^n$ qui nous permet ensuite d'établir les bonnes propriétés de cette dimension mixte (qui est par exemple invariante par bijection sous-analytique, cf. proposition 4.2.14(3)). La formule principale que nous prouvons est la suivante :

Proposition 7. *Si $f : X \to Y$ est une application sous-analytique entre parties sous-analytiques de $((k^{alg})^{\circ})^* \Gamma^*$, alors*

$$\dim(f(X)) \leq \max_{d \geq 0} \left((\dim(X)) + (-d, d)\right). \tag{4}$$

Par définition, $\dim(X)$ étant une partie finie de $\mathbb{N}^2$, dans la formule ci-dessus max signifie en fait $\cup$.

Annexes

À force d'utiliser le théorème d'élimination des quantificateurs pour ACVF, on finit par se demander : *au fait, quels sont les analogues Archimédiens ?* Bien sûr le théorème de Tarski-Seidenberg est le bon analogue sur $\mathbb{R}$, mais quid de $\mathbb{C}$, qui est le seul corps Archimédien complet algébriquement clos ? Disons qu'une partie S de $\mathbb{C}^n$ est **semi-algébrique complexe** si c'est une combinaison booléenne finie d'inégalités de la forme $|f| \leq |g|$ avec $f, g \in \mathbb{C}[T_1, \ldots, T_n]$. Nous montrons que les ensembles semi-algébriques complexes ne sont pas stables par projection. Bien qu'il s'agisse d'une simple remarque, ce résultat, qui à notre connaissance ne figure pas dans la littérature, nous a semblé mériter d'être explicité.

Questions en suspens

Théorème de descente. On peut se demander s'il existe des résultats de descente pour les partie semi-analytiques (resp., sous-analytiques (surconvergents)...). Précisément, si $k \to K$ est une extension non archimédienne algébrique, X un espace k-affinoïde, et $S \subset X$ une partie de X telle que $S_K \subset X_K$ soit une partie semi-analytique (resp. ...) de X_K. Alors est-ce que S est semi-analytique (resp. ...) de X ?

Faisceaux constructibles. On peut chercher à généraliser les résultats de finitude du chapitre 2 dans deux directions.

Dans le cas absolu on peut espérer

Conjecture 1. *Si k est algébriquement clos, $\ell \neq car(\tilde{k})$, X est un espace k-affinoïde et S une partie sous-analytique localement fermée de X alors les groupes $H^q_c((X,S), \mathbb{Z}/\ell^n\mathbb{Z})$ sont finis.*

En adaptant les arguments de la preuve de 2.4.1, ce résultat serait une conséquence de

Conjecture 2. *Si k est algébriquement clos, $\ell \neq car(\tilde{k})$, X est un espace k-quasi-affinoïde*[9] *alors les groupes $H^q_c(X, \mathbb{Z}/\ell^n\mathbb{Z})$ sont finis.*

Le corollaire 3.1.2 de [Ber13], va dans ce sens.

Dans le cas relatif, nous espérons convaincre le lecteur de ce travail que si l'on cherche à définir la plus petite classe de faisceaux stable par $Rf_!$ pour les morphismes entre espaces affinoïdes f, un bon candidat pourrait être de définir un faisceau sous-analytique comme un faisceau étale sur X pour lequel il existe un recouvrement fini $\{S_i\}$ de X par des parties sous-analytiques localement fermées de X tel que $F_{|S_i}$ soit localement constant sur le germe (X, S_i).

Conjecture 3. *Soit $f : X \to Y$ un morphisme d'espaces k-affinoïdes, F un faisceau sous-analytique sur X. Alors $R^q f_!(F)$ est un faisceau sous-analytique sur Y.*

Un tel résultat impliquerait les conjectures précédentes pour le cas absolu. Signalons que dans cette direction, une idée pourrait être d'adapter la preuve de [Hub98b].

Espaces quasi-affinoïdes façon Berkovich ou Huber. Comprendre les objets obtenus en appliquant les foncteurs $\mathcal{M}$ et Spa aux algèbres quasi-affinoïdes (voir [Kap12, section 4] pour des remarques analogues). Quand k est de valuation discrète, les algèbres quasi-affinoïdes sont fortement noethériennes, de sorte que Spa nous donne bien un espace adique [Hub94, théorème 2.2], et on dispose donc du formalisme de la cohomologie étale des espaces adiques [Hub96]. Ce genre de considération pourrait être une des manières de montrer des résultats tels que la conjecture 3.

9. Voir la section 0.5.2 pour la définition d'un espace quasi-affinoïde.

Chapitre 0

Préliminaires

0.1 Géométrie rigide, espaces de Berkovich

Un corps normé est un corps k muni d'une valuation $|\cdot| : k \to \mathbb{R}_+$ telle que $|xy| = |x| \cdot |y|$, $|x+y| \leq |x| + |y|$ et $|x| = 0 \Leftrightarrow x = 0$. S'il vérifie de plus la condition plus forte $|x+y| \leq \max(|x|, |y|)$ on dit que la valuation est non archimédienne. C'est équivalent au fait que $|n| \leq 1$ pour tout $n \in \mathbb{Z}$. Dans le cas contraire on dit que le corps k est Archimédien. A isomorphisme près tout corps Archimédien est un sous corps de $\mathbb{C}$ muni de la norme $\|\cdot\|^s$ où $s \in]0,1]$ et $\|\cdot\|$ est la norme euclidienne [Neu99, II.4.2]. En particulier, un corps Archimédien complet est $\mathbb{R}$ ou $\mathbb{C}$, équipée de la norme $\|\cdot\|^s$. Des pans entiers des mathématiques sont consacrés à la géométrie analytique sur ces deux corps. On appellera **corps non archimédien** un corps k complet pour une valuation non archimédienne [1]. Ce sont donc tous les corps complets autres que $\mathbb{R}$ et $\mathbb{C}$. Dans cette section, on va expliquer différentes approches pour faire de la géométrie analytique sur un tel corps k.

On présentera d'abord l'approche de la géométrie rigide initiée par Tate [Tat71] et dont un développement très complet est donné dans [BGR84]. Les espaces ainsi construits sont appelés espaces rigides et forment une catégorie notée Rig_k. Ensuite nous présenterons l'approche de Berkovich. Les espaces qu'il construit sont appelés espaces k-analytiques. Huber a étudié d'autres espaces, les espaces adiques. Précisons qu'il y a deux foncteurs pleinement fidèles (rappelons que l'on suppose la valuation non triviale) :

$$\left\{ \begin{array}{c} \text{espaces strictement} \\ k\text{-analytiques} \\ \text{Hausdorff} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{espaces} \\ \text{rigides sur } k \end{array} \right\} \xrightarrow{r_k} \left\{ \text{espaces adiques} \right\}$$

Ces inclusions sont bien expliquées dans [Sch12, section 2]. Mentionnons également une approche utilisant les schémas formels initiée par Raynaud [Ray74] et dont on peut trouver les fondements dans [BL93a, BL93b, BLR95a, BLR95b, Abb11].

0.1.1 Géométrie rigide

Soit $f = \sum_{\nu \in \mathbb{N}^n} a_\nu X^\nu$ une série formelle de $k[[X_1, \ldots, X_n]]$. Il y a équivalence entre

– Pour tout $x = (x_1, \ldots, x_n) \in (k^\circ)^n$, $f(x)$ converge.

1. Sur un corps quelconque, l'application $x \mapsto |x| = 1$ pour $x \neq 0$ définit une valuation appelée valuation triviale. Considérer des espaces de Berkovich sur ces corps s'est avéré un des atouts des espaces de Berkovich ([Thu07] par exemple), mais sauf mention contraire, nous supposerons que la valuation sur k est non triviale.

– Pour tout $x = (x_1, \ldots, x_n) \in ((k^{\mathrm{alg}})^\circ)^n$, $f(x)$ converge.
– $\lim\limits_{\substack{\nu \in \mathbb{N}^n \\ |\nu| \to \infty}} |a_\nu| = 0$.

La k-algèbre des séries vérifiant ces conditions est appelée l'algèbre de Tate et notée $k\langle X_1, \ldots, X_n\rangle$. Berkovich la note plutôt $k\{X_1, \ldots, X_n\}$. C'est un anneau noethérien, qui est même excellent, pour lequel on dispose d'un Nullstellensatz : l'application

$$\begin{array}{ccc} ((k^{\mathrm{alg}})^\circ)^n & \to & \mathrm{Max}(k\langle X_1, \ldots, X_n\rangle) \\ x & \mapsto & \mathfrak{m}_x = \{f \in k\langle X_1, \ldots, X_n\rangle \mid f(x) = 0\} \end{array}$$

est surjective et identifie $\mathrm{Max}(k\langle X_1, \ldots, X_n\rangle)$ avec les orbites de $((k^{\mathrm{alg}})^\circ)^n$ sous l'action de $\mathrm{Gal}(k^{sep}/k)$.

On munit $k\langle X_1, \ldots, X_n\rangle$ de la norme

$$\begin{array}{cccc} \|\cdot\| : & k\langle X_1, \ldots, X_n\rangle & \to & \mathbb{R}_+ \\ & f = \sum\limits_{\nu \in \mathbb{N}^n} a_\nu X^\nu & \mapsto & \max\limits_{\nu \in \mathbb{N}^n} |a_\nu| \end{array}$$

qui fait de $k\langle X_1, \ldots, X_n\rangle$ une k-algèbre de Banach. Une algèbre k-affinoïde $\mathcal{A}$ est une k-algèbre de Banach isomorphe à un quotient $k\langle X_1, \ldots, X_n\rangle/I$, la norme sur $\mathcal{A}$ étant la norme quotient héritée de $k\langle X_1, \ldots, X_n\rangle$ (il s'agit bien d'une norme, car on peut montrer que tous les idéaux de $k\langle X_1, \ldots, X_n\rangle$ sont fermés). Comme $k\langle X_1, \ldots, X_n\rangle$ est noethérienne, on peut supposer que I est engendrée par un nombre fini d'éléments, $f_1, \ldots, f_m \in k\langle X_1, \ldots, X_n\rangle$, et d'après le Nullstellensatz, on a une correspondance bijective entre $\mathrm{Max}(\mathcal{A})$ et les $\mathrm{Gal}(k^{sep}/k)$-orbites de l'ensemble $\{x \in ((k^{\mathrm{alg}})^\circ)^n \mid f_i(x) = 0,\ i = 1 \ldots, m\}$. Si $r = (r_1, \ldots, r_n) \in \sqrt{|k^\times|}^n$ alors on note $k\{r^{-1}X\} := k\{r_1^{X_1}, \ldots, r_n^{-1}X_n\}$ l'algèbre des séries formelles $\sum a_\nu X^\nu$ telles que $|a_\nu| r^\nu \to 0$. C'est une algèbre k-affinoïde [BGR84, 6.1.5.4]. Plus généralement, si $\mathcal{A}$ est une algèbre k-affinoïde et $r = (r_1, \ldots, r_n) \in \sqrt{|k^\times|}^n$ un polyrayon, on notera

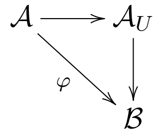
$$\mathcal{A}\{r^{-1}X\} = \mathcal{A}\{r_1^{-1}X_1, \ldots, r_n^{-1}X_n\}$$

l'algèbre des séries $\sum_\nu a_\nu X^\nu$ telles que $|a_\nu| r^\nu \to 0$, et c'est donc aussi une algèbre affinoïde.

Définition 0.1.1. *Si $\mathcal{A}$ est une algèbre k-affinoïde, on note $Sp(\mathcal{A}) := Max(\mathcal{A})$ l'ensemble des idéaux maximaux de $\mathcal{A}$, qu'on appelle l'espace affinoïde associé à $\mathcal{A}$.*

On aimerait que X soit une variété analytique sur k, et que $\mathcal{A}$ soit l'ensemble des fonctions analytiques sur X. Si $\varphi : \mathcal{A} \to \mathcal{B}$ est un morphisme entre algèbres k-affinoïdes, on obtient une application $\mathrm{Sp}(\varphi) : \mathrm{Sp}(\mathcal{B}) \to \mathrm{Sp}(\mathcal{A})$.

Définition 0.1.2. *Une partie $U \subset X$ est un domaine affinoïde de X s'il existe une algèbre k-affinoïde $\mathcal{A}_U$, un morphisme d'algèbre affinoïdes $\mathcal{A} \to \mathcal{A}_U$ tel que pour tout morphisme d'algèbres affinoïdes $\varphi : \mathcal{A} \to \mathcal{B}$ vérifiant $im(Sp(\varphi)) \subset U$ il existe un unique morphisme $\mathcal{A}_U \to \mathcal{B}$ rendant commutatif*



Dans ce cas, le morphisme induit $\mathrm{Sp}(\mathcal{A}_U) \to \mathrm{Sp}(\mathcal{A})$ induit une bijection entre $\mathrm{Sp}(\mathcal{A}_U)$ et U, et factorise de manière unique tous les morphismes d'espaces affinoïdes vers X d'image incluse dans U. Ainsi, $\mathcal{A}_U$ est unique, à un unique isomorphisme près, et on la note aussi $\mathcal{O}_X(U)$.

Si U, V sont des domaines affinoïdes de X, $U \cap V$ aussi ; et si $U \subset V$, alors U est également un domaine affinoïde de V.

Soit $f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathcal{A}$ et $p_1, \ldots, p_m, q_1, \ldots, q_n \in \sqrt{|k^\times|}$. Avec des inégalités faisant intervenir les f_i, g_j, on définit des parties de X, qui s'avèrent être des domaines affinoïdes :

- $\{x \in X \mid |f_i(x)| \leq p_i,\ i = 1 \ldots m\}$ est appelé un domaine de Weierstrass.
- $\{x \in X \mid |f_i(x)| \leq p_i,\ |g_j(x)| \geq q_j,\ i = 1 \ldots, m,\ j = 1, \ldots, n\}$ est appelé un domaine de Laurent.
- Si les f_i n'ont pas de zero commun, $\{x \in X \mid |f_i(x)| \leq |f_1(x)|,\ i = 1, \ldots, n\}$ est un domaine rationnel .

Un domaine de Weierstrass est un domaine de Laurent qui est lui même un domaine rationnel, et ce sont tous des domaines affinoïdes [Bos, 1.6.11], [Ber90, 2.2.2]. Par exemple, l'algèbre affinoïde de $\{x \in X \mid |f_i(x) \leq p_i,\ i = 1 \ldots m\}$ est isomorphe à $\mathcal{A}\{p_1^{-1}T_1, \ldots, p_m^{-1}T_m\}/(f_i - T_i)$.

Théorème 0.1.3 (Théorème d'acyclicité de Tate)**.** *Soit $\{U_i\}_{1 \leq i \leq n}$ un recouvrement fini de X par des domaines affinoïdes. La suite*

$$\mathcal{A} \to \prod_{i=1}^{n} \mathcal{A}_{U_i} \to \prod_{1 \leq i < j \leq n} \mathcal{A}_{U_i \cap U_j}$$

est exacte.

Sur X, on appelle topologie faible la G-topologie dont les ouverts admissibles sont les domaines affinoïdes, et un recouvrement admissible de U est un recouvrement fini par des domaines affinoïdes. La propriété universelle des domaines affinoïdes fait de $\mathcal{O}_X$ un préfaisceau pour cette topologie, qui est même un faisceau grâce au théorème d'acyclicité. En revanche, $\mathcal{O}_X$ n'est pas un faisceau pour la topologie usuelle héritée de celle de k. Par exemple, si l'on suppose k algébriquement clos, on peut recouvrir le disque unité fermé par les boules ouvertes de rayon 1, qui sont en nombre infini, indexées par $\tilde{k}$, et disjointes. Si l'on recolle des fonctions constantes sur chacune de ces boules, on obtient une fonction qui ne provient pas de $k\{T\}$.

0.1.2 Espaces de Berkovich

V. Berkovich associe à une algèbre k-affinoïde l'espace $X = \mathcal{M}(\mathcal{A})$ des semi-normes multiplicatives bornées sur $\mathcal{A}$. Précisément ce sont les applications $|\cdot| : \mathcal{A} \to \mathbb{R}_+$ telles que $|f+g| \leq |f| + |g|$, $|fg| = |f||g|$ (semi-norme) et $|f| \leq \|f\|$ (bornée). Si $x \in \mathcal{M}(\mathcal{A})$, on note $|f(x)| := x(f)$. On munit X de la topologie la plus faible rendant continue les applications $|f| : X \to \mathbb{R},\ x \mapsto |f(x)|$. Dit autrement on considère sur X la topologie engendrée par les ouverts $\{x \in X \mid a < |f(x)| < b\}$ pour $f \in \mathcal{A},\ a, b \in \mathbb{R}$. Le fait remarquable est que X est un espace compact [Ber90, 1.2.1], et localement connexe par arcs [Ber90, 3.2.1]. De manière naturelle, $\mathcal{M}(\mathcal{A})$ contient $\mathrm{Sp}(\mathcal{A})$: si $x \in \mathrm{Sp}(\mathcal{A})$ correspond à un idéal maximal $\mathfrak{m}$, alors $K := \mathcal{A}/\mathfrak{m}$ est une extension finie de k d'après le Nullstellensatz, la norme de k se prolonge donc à K de manière unique, et cela définit par composition avec $\mathcal{A} \to \mathcal{A}/\mathfrak{m}$ une semi-norme multiplicative sur $\mathcal{A}$. On appelle un tel point de $\mathcal{M}(\mathcal{A})$ un **point rigide**. On notera l'espace affinoïde correspondant à la boule unité fermée

$$\mathbb{B} := \mathcal{M}(k\langle T\rangle)$$

et plus généralement le polydisque unité :

$$\mathbb{B}^n := \mathcal{M}(k\langle T_1, \ldots, T_n\rangle.$$

Parfois, pour éviter toute ambiguïté sur le corps de base, on les notera $\mathbb{B}_k$ et $\mathbb{B}_k^n$.

Si $x \in X$, notons $\mathfrak{p}_x := \{f \in \mathcal{A} \mid |f(x)| = 0\}$. C'est un idéal premier de $\mathcal{A}$ et la semi-norme x induit une norme sur $\mathcal{A}/\mathfrak{p}_x$ (car $\mathfrak{p}_x$ est fermé), et donc sur son corps des fractions $\mathrm{Frac}(\mathcal{A}/\mathfrak{p}_x)$. On note $\mathcal{H}(x)$ son complété, qu'on appelle le **corps résiduel complété** de x. Remarquons que x est un point rigide si et seulement si $\mathcal{H}(x)$ est une extension finie de k.

Par construction, on obtient alors un morphisme (qui est continu) $\chi : \mathcal{A} \to \mathcal{H}(x)$ obtenu comme la composée $\mathcal{A} \to \mathcal{A}/\mathfrak{p}_x \to \mathcal{H}(x)$, de sorte que si $f \in \mathcal{A}$, $|f(x)| = |\chi(f)|_{\mathcal{H}(x)}$. Cela justifie que l'on note souvent $f(x) = \chi(f)$.

Réciproquement, appelons **caractère** de $\mathcal{A}$ un morphisme d'anneaux $\chi : \mathcal{A} \to L$ où L est une extension non archimédienne de k et où χ est bornée. A un tel caractère, on associe une semi-norme multiplicative bornée :

$$\begin{array}{cccc} |\cdot|_\chi : & \mathcal{A} & \to & \mathbb{R} \\ & f & \mapsto & |\chi(f)|_L \end{array} \tag{1}$$

qui est donc un élément de $\mathcal{M}(\mathcal{A})$. Deux caractères $\chi_1 : \mathcal{A} \to L_1$, $\chi_2 : \mathcal{A} \to L_2$ sont dits équivalents s'il existe un troisième caractère $\chi : \mathcal{A} \to L$ et des plongements de corps non archimédiens $L_1 \to L$ et $L_2 \to L$ rendant commutatifs

$$\begin{array}{ccccc} & & \mathcal{A} & & \\ & \swarrow^{\chi_1} & \downarrow^{\chi} & \searrow^{\chi_2} & \\ L_1 & \longrightarrow & L & \longleftarrow & L_2 \end{array}$$

Cela définit bien une relation d'équivalence sur l'ensemble des caractères, et la formule (1) induit une bijection :

$$\begin{array}{ccc} \{\text{Caractères sur } \mathcal{A}\}/\sim & \xrightarrow{\phi} & \mathcal{M}(\mathcal{A}) \\ \chi & \mapsto & |\cdot|_\chi \end{array}$$

Démonstration. Cette application est surjective car si $x \in \mathcal{M}(\mathcal{A})$, le caractère $\mathcal{A} \to \mathcal{H}(x)$ est un antécédent de x par ϕ. De plus, si $\chi : \mathcal{A} \to L$ est un caractère, $\ker(\chi) := \{f \in \mathcal{A} \mid \chi(f) = 0\}$ est un idéal premier de $\mathcal{A}$ et on obtient des factorisations

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \mathcal{A}/(\ker(\chi)) & \longrightarrow & \mathrm{Frac}(\mathcal{A}/\ker(\chi)) \\ & {}_{\chi}\searrow & \downarrow & \swarrow & \\ & & L & & \end{array}$$

Si on note K le complété de $\mathrm{Frac}(\mathcal{A}/\ker(\chi))$ pour sa norme induite, alors K est isomorphe à $\mathcal{H}(x)$, où x est l'image par ϕ de χ. Ainsi, par propriété universelle de la complétion, on a une factorisation

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{H}(x) \\ & {}_{\chi}\searrow & \downarrow \\ & & L \end{array}$$

et cela prouve que ϕ est injective. $\square$

Une dernière remarque. Si $k \to L$ est une extension non archimédienne, on a une bijection

$$\left\{ \text{Les caractères} \quad \chi : k\langle X_1, \ldots, X_n \rangle \to L \right\} \leftrightarrow (L^\circ)^n.$$

A un point $(x_1, \ldots, x_n) \in (L^\circ)^n$ est associé le caractère

$$\begin{array}{ccc} k\langle X_1, \ldots, X_n\rangle & \to & L \\ f & \mapsto & f(x) \end{array}$$

et réciproquement, au caractère χ on fait correspondre le n-uplet $(\chi(X_i))_i$. On en déduit une application $\pi : (L^\circ)^n \to \mathbb{B}^n_k$. Cette application n'est en général pas surjective.

Si $L = k^{\mathrm{alg}}$ par exemple, son image est l'ensemble des points rigides de $\mathbb{B}^n$ et donc π n'est pas surjective. Cependant il existe des gros corps L pour lesquels π est surjective.

De même π n'est pas injective en général. Si $L = k^{\mathrm{alg}}$ toujours, les éléments d'une même orbite sous $\mathrm{Gal}(k^{sep}/k)$ ont même image. On verra même en 3.1.13 des points de $(L^\circ)^n$ où π est localement constante [2].

On définit comme pour les espaces rigides la notion de domaine affinoïde de X, et en utilisant le théorème d'acyclicité de Tate, l'espace affinoïde X est muni d'un faisceau $\mathcal{O}_X$ qui fait de X est espace localement annelé [Ber90, 2.3].

Recoller les espaces affinoïdes n'est alors pas évident : les espaces affinoïdes étant compacts, ils ne seront en général pas ouverts dans leur espace ambiant. Dans [Ber90, Chapitre 3] un procédé de globalisation est expliqué. Les espaces ainsi obtenus sont les **bons espaces k-analytiques**. Dans [Ber93, 1], la catégorie plus générale des **espaces k-analytiques** est définie. Dans ses articles, Berkovich parle d'espaces k-analytiques. Certains auteurs parlent d'**espaces de Berkovich**, mais il s'agit bien des mêmes objets. Signalons que les bons espaces k-analytiques sont les espaces k-analytiques tels que tout point x ait une base de voisinages affinoïdes. C'est le cas par exemple quand X est affinoïde. Même si X n'est pas bon, tout point admet un voisinage qui est une union finie de domaines affinoïdes.

Pour finir signalons que l'un des intérêts des espaces de Berkovich est de pouvoir utiliser des rayons $r \notin \sqrt{|k^\times|}$, ce qui permet notamment de traiter le cas où la valuation sur k est triviale. Nous n'utiliserons pas cette propriété sauf mention du contraire. Nos espaces seront strictement k-analytiques en général.

0.1.3 Foncteur d'analytification

Si $\mathcal{X}$ est une k-schéma de type fini, on peut lui associer un espace k-analytique $\mathcal{X}^{\mathrm{an}}$ [Ber90, 3.4.1] pour lequel les théorèmes de type GAGA sont valables [Ber90, 3.4.9–3.4.14]. Si $\mathcal{X} = \mathrm{Spec}(A)$, alors $\mathcal{X}^{\mathrm{an}}$ correspond aux semi-normes multiplicatives

$$|\cdot| : A \to \mathbb{R}_+$$

qui prolongent la norme de k. En particulier, $(\mathbb{A}^n_k)^{\mathrm{an}}$ correspond aux semi-normes $|\cdot| : k[X_1, \ldots, X_n] \to \mathbb{R}_+$ prolongeant la norme de k, et s'identifie à l'union croissante $\cup_{r>0}\mathbb{B}^n_r$.

Si $A = k[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$,

$$\mathcal{X}^{\mathrm{an}} = \{x \in (\mathbb{A}^n_k)^{\mathrm{an}} \mid f_i(x) = 0,\ i = 1 \ldots, m\}$$

et de manière équivalente,

$$\mathcal{X}^{\mathrm{an}} = \cup_{r>0}\mathcal{M}(k\{r^{-1}X_1, \ldots, r^{-1}X_n\}/(f_i).$$

Plus généralement, si $\mathcal{A}$ est une k-algèbre affinoïde, $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$ et $\mathcal{Y}$ un $\mathcal{X}$-schéma de type fini, on peut de manière analogue, lui associer un espace k-analytique au dessus de $X = \mathcal{M}(\mathcal{A})$ [Ber93, 2.6.1]. Par exemple, $(\mathcal{X} \times_k \mathbb{A}^n)^{\mathrm{an}} \simeq X \times_k (\mathbb{A}^n_k)^{\mathrm{an}}$.

2. Nous voulons dire par cela qu'il existe un point $x \in (L^\circ)^n$, et V un voisinage de x tels que $\pi_{|V}$ est constante.

0.1.4 Intérieur, bord

L'un des avantages des espaces de Berkovich est de matérialiser de manière tangible des phénomènes de bord. Soit $\varphi : \mathcal{A} \to \mathcal{B}$ un morphisme d'algèbres k-affinoïdes qui induit un morphisme d'espaces k-affinoïdes $f : X \to Y$. Alors il existe un épimorphisme $\pi : \mathcal{A}\{r^{-1}T\} \twoheadrightarrow \mathcal{B}$. Par exemple, si $\psi : k\{T_1, \dots, T_n\} \twoheadrightarrow \mathcal{B}$ est un épimorphisme, $\varphi\widehat{\otimes}\psi\mathcal{A}\{T_1, \dots, T_n\} \twoheadrightarrow \mathcal{B}$ est bien un épimorphisme. Il induit une immersion fermée ι d'espaces k-affinoïdes

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & X \times \mathbb{B}^n \\ & \underset{\varphi}{\searrow} & \downarrow \\ & & X \end{array}$$

où la flèche verticale est la première projection.

Définition 0.1.4. *L'intérieur relatif de φ, noté Int(Y/X) est la partie de Y formée des points y tel qu'il existe un épimorphisme $\pi : \mathcal{A}\{T_1, \dots, T_n\} \twoheadrightarrow \mathcal{B}$ pour lequel $\iota(y)$ tombe dans $(\overset{\circ}{\mathbb{B}})^n$, i.e. $|T_i(y)| < 1$ pour tout i. Le bord relatif, noté $\partial(Y/X)$ est le complémentaire de Int(Y/X).*

Proposition 0.1.5. *[Ber90, 2.5.9] Soit $\psi : Z \to Y$ et $\varphi : Y \to X$ deux morphismes d'espaces k-affinoïdes, alors im$(\psi) \subset$ Int(Y/X) si et seulement si il existe une factorisation*

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & X \times \mathbb{B}^n \\ & \underset{\varphi}{\searrow} & \downarrow \\ & & X \end{array}$$

où ι est une immersion fermée telle que $\iota(im(\psi)) \subset X \times (\overset{\circ}{\mathbb{B}})^n$. $(s_1, \dots, s_n) < (r_1, \dots, r_n)$.

Signalons également que si Y est un domaine affinoïde de X et $\varphi : Y \to X$ le morphisme associé, Int(Y/X) correspond à l'intérieur topologique de Y dans X [Ber90, 2.5.13]. Quand $X = \mathcal{M}(k)$, on écrit Int(Y) à la place de Int$(Y/\mathcal{M}(k))$ qu'on appelle simplement l'intérieur de Y, et $\partial(Y) := \partial(Y/\mathcal{M}(k))$ le bord de Y. Les énoncés précédents deviennent :

Définition 0.1.6. *Un point $y \in$ Int(Y) si il existe une immersion fermée $\iota : Y \to \mathbb{B}_{\underline{r}}$ telle que $\iota(y) \in \mathbb{B}_{\underline{s}}$ pour un polyrayon $\underline{s} < \underline{r}$.*

Proposition 0.1.7. *Si $\psi : Z \to Y$ est un morphisme d'espaces k-affinoïdes, im$(\psi) \subset$ Int(Y) si et seulement si il existe une immersion fermée $\iota : Y \to \mathbb{B}_{\underline{r}}$ telle que $\iota(im(\psi)) \subset \mathbb{B}_{\underline{s}}$ pour un polyrayon $\underline{s} < \underline{r}$.*

0.2 Théorie de Weierstrass

La théorie de Weierstrass, que l'on va présenter ici, est un des outils qui permet d'établir les bonnes propriétés algébriques des algèbres k-affinoïdes telles que le lemme de normalisation de Noether, la noetheriannité, le Nullstellensatz... On renvoie à [BGR84, 5.2] pour un traitement complet, et [Bos, 1.2] [FvdP04, 3.1,3.2] pour une présentation efficace.

On dispose d'une application de réduction

$$\begin{array}{ccc} k^\circ\langle X_1, \dots, X_n\rangle & \to & \tilde{k}[X_1, \dots, X_n] \\ f = \sum\limits_{\nu \in \mathbb{N}^n} a_\nu X^\nu & \mapsto & \sum_{\nu \in \mathbb{N}^n} \widetilde{a_\nu} X^\nu \end{array}$$

Si $f \in k\langle X_1, \dots, X_n\rangle$ et $\|f\| = 1$, alors f est inversible si et seulement si $\tilde{f} \in \tilde{k}^*$. De manière générale, $f = \sum_\nu a_\nu X^\nu \in k\langle X_1, \dots, X_n\rangle$ est inversible si et seulement si $|a_0| > |a_\nu|$ pour tout $\nu \in \mathbb{N}^n \setminus (0, \dots, 0)$, et dans ce cas, si $x \in \mathbb{B}^n$, $|f(x)| = |a_0| = \|f\|$, de sorte que $x \in \mathbb{B}^n \mapsto |f(x)|$ est constante égale à $\|f\|$.

Définition 0.2.1. *Soit $s \in \mathbb{N}$ et $g \in k\langle X_1, \dots, X_n\rangle$ qu'on écrit sous la forme $g = \sum_{j=0}^{\infty} g_j X_n^j$ avec pour tout j, $g_j \in k\langle X_1, \dots, X_{n-1}\rangle$. On dit que g est X_n-distinguée d'ordre s si g_s est inversible, pour $j > s$, $\|g_j\| < \|g_s\|$ et pour $j \leq s$, $\|g_j\| \leq \|g_s\|$.*

On dit que g est X_n-distinguée s'il existe s tel que g est X_n-distinguée d'ordre s.

Définition 0.2.2. *Soit $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{N}$. Alors l'application*

$$\begin{array}{rccc} \sigma : & k\langle X_1, \dots, X_n\rangle & \to & k\langle X_1, \dots, X_n\rangle \\ & X_i & \mapsto & \begin{cases} X_i + X_n^{\alpha_i} & si\ i < n \\ X_n & si\ i = n \end{cases} \end{array}$$

*définit un automorphisme de $k\langle X_1, \dots, X_n\rangle$ que l'on appelle **automorphisme de Weierstrass**.*

Il faut vérifier d'une part qu'il existe bien un unique morphisme de k-algèbres $k\langle X_1, \dots, X_n\rangle \to k\langle X_1, \dots, X_n\rangle$ vérifiant ces conditions, et d'autre part que σ est bien inversible. Son inverse est donné par la formule $X_i \mapsto -X_i^{\alpha_i}$ pour $i < n$ et $X_n \mapsto X_n$.

Lemme 0.2.3 ([Bos, 1.2.7])**.** *Soit $f_1, \dots, f_r \in k\langle X_1, \dots, X_n\rangle$. Alors il existe un automorphisme de Weierstrass σ tel que pour tout i $\sigma(f_i)$ est X_n distingué.*

Théorème 0.2.4 (Division de Weierstrass, [Bos, 1.2.8])**.** *Soit $g \in k\langle X_1, \dots, X_n\rangle$ une série X_n-distinguée d'ordre s. Alors pour tout $f \in k\langle X_1, \dots, X_n\rangle$, il existe un unique couple $(q, R) \in k\langle X_1, \dots, X_n\rangle \times k\langle X_1, \dots, X_{n-1}\rangle[X_n]$ tel que*

$$\begin{aligned} \deg(R) &< s \\ f &= gq + R \\ \|f\| &= \max(\|q\|, \|R\|). \end{aligned}$$

On obtient comme corollaire :

Théorème 0.2.5 (Théorème de préparation de Weierstrass [Bos, 1.2.9])**.** *Si $g \in k\langle X_1, \dots, X_n\rangle$ est X_n-distinguée d'ordre s, il existe un unique couple $(e, w) \in k\langle X_1, \dots, X_n\rangle \times k\langle X_1, \dots, X_{n-1}\rangle[X_n]$ tel que e soit inversible et w est un polynôme unitaire de degré s avec*

$$g = ew.$$

0.3 Ensembles semi-analytiques, semi-algébriques et variantes locales

0.3.1 Ensembles semi-analytiques

On fixe une algèbre k-affinoïde $\mathcal{A}$, et l'on note $X := \mathcal{M}(\mathcal{A})$ l'espace k-affinoïde associé.

Définition 0.3.1. *Un ensemble $S \subset X$ est **semi-analytique** si c'est une combinaison booléenne de parties du type*

$$\{x \in X \mid |f(x)| \leq |g(x)|\}$$

où $f, g \in \mathcal{A}$.

On définit de manière analogue une partie semi-analytique de $\mathrm{Sp}(\mathcal{A})$ en remplaçant X par $\mathrm{Sp}(\mathcal{A})$. On expliquera dans la section 1.2.4 qu'il y a une correspondance entre les parties semi-analytiques de X et celles de $\mathrm{Sp}(\mathcal{A})$. Signalons aussi que si l'on note X_{red} l'espace affinoïde réduit associé à X (dont l'algèbre affinoïde est $\mathcal{A}^{red}$), les parties semi-analytiques de X et de X_{red} sont en bijection.

Il est facile de voir que si $f, g \in \mathcal{A}$, les ensembles $\{x \in X \mid |f(x)| < |g(x)|\}$, $\{x \in X \mid |f(x)| = |g(x)|\}$ et $\{x \in X \mid f(x) = 0\}$ sont semi-analytiques. Si $r, s \in \sqrt{|k^\times|}$, alors $\{x \in X \mid r|f(x)| \leq s|g(x)|\}$ est aussi semi-analytique : si on prend $n \in \mathbb{N}$ tel que $r^n, s^n \in |k^*|$, cette partie s'écrit aussi $\{x \in X \mid r^n|f^n(x)| \leq s^n|g^n(x)|\}$. Prendre des valeurs quelconques $r \in \mathbb{R}_+$ ne marchera cependant pas [3].

Remarquons que X et $\emptyset$ sont semi-analytiques. On voit facilement que les ensembles Zariski-fermés de X sont semi-analytiques ainsi que les domaines rationnels. Ayons en tête le résultat fondamental suivant

Théorème 0.3.2 (Théorème de Gerritzen-Grauert [Bos, 1.8.12], [BGR84, 7.3.5] et pour une approche avec des espaces de Berkovich [Tem05])**.** *Tout domaine affinoïde de X est réunion finie de domaines rationnels de X.*

Cela implique que tout domaine affinoïde est semi-analytique.

Définition 0.3.3. *Une partie S de X est* ***semi-analytique basique*** *si elle peut s'écrire*

$$S = \bigcap_{i=1}^{n} \{x \in X \mid |f_i(x)| \bowtie_i |g_i(x)|\}$$

où pour tout i, $f_i, g_i \in \mathcal{A}$ et $\bowtie_i \in \{\leq, <\}$. Dit autrement, S est une intersection finie d'inégalités de la forme $|f| \leq |g|$ et $|f| < |g|$.

Lemme 0.3.4. *Toute partie semi-analytique de X est réunion finie de parties semi-analytiques basiques de X.*

Nous utiliserons fréquemment sans le préciser le résultat suivant qui découle de la définition :

Lemme 0.3.5. *Soit $f : Y \to X$ un morphisme d'espaces k-affinoïdes, et $S \subset X$ une partie semi-analytique de X. Alors $f^{-1}(S)$ est une partie semi-analytique de Y.*

En particulier, si U est un domaine affinoïde de X et $S \subset X$ une partie semi-analytique de X, alors $S \cap U$ est une partie semi-analytique de X.

En revanche, si S est une partie semi-analytique de Y, $f(S)$ ne sera pas semi-analytique en général. Il suffit de prendre $0 < r < 1$ avec $r \in \sqrt{|k^\times|}$, puis une fonction $f \in k\{r^{-1}T\}$ de rayon de convergence exactement r, et telle que $\|f\| < 1$. On considère alors le morphisme d'espaces affinoïdes

$$\begin{array}{cccc} \varphi : & \mathbb{B}_r & \to & \mathbb{B}^2 \\ & x & \mapsto & (x, f(x)) \end{array}$$

Si l'on prend tout simplement $S = \mathbb{B}_r$ (qui est bien une partie semi-analytique de $\mathbb{B}_r$), alors $\varphi(\mathbb{B}_r)$ n'est pas une partie semi-analytique de $\mathbb{B}^2$ (cet exemple sera expliqué en détail dans la section 1.2.2).

3. Par exemple, soit $r \in \mathbb{R} \setminus \sqrt{|k^\times|}$ et supposons qu'il existe $x \in X$ tel que $|f(x)| = r$. Posons alors $S = \{x \in X \mid |f(x)| \leq r\}$. Si S était semi-analytique, alors d'après le théorème d'élimination des quantificateurs de Lipshitz que nous présenterons en 0.5.1, $|S|$, l'ensemble des normes des éléments de S, devrait être définissable sur $\sqrt{|k^\times|}$. Mais le sup de cet ensemble est r qui n'appartient pas à $\sqrt{|k^\times|}$. Donc S n'est pas semi-analytique.

C'est ce phénomène qui explique la figure 1 page 13 et que l'on doive prendre en considération toutes les autres classes de parties.

Nous voudrions insister sur deux bonnes raisons de considérer les parties semi-analytiques si l'on s'intéresse aux images de morphismes entre espaces affinoïdes.

La première est qu'ils sont l'analogue analytique des parties semi-algébriques réelles qui ont de bonnes propriétés de stabilité par image.

La seconde est la suivante. Si $X = \mathcal{M}(\mathcal{A})$, et $S = \{x \in X \mid |f(x)| \leq |g(x)|\}$ avec $f, g \in \mathcal{A}$, considérons $\mathcal{B} := \mathcal{A}\{T\}/(f - Tg)$ et $Y = \mathcal{M}(\mathcal{B})$. Le morphisme naturel d'algèbres affinoïdes $\mathcal{A} \to \mathcal{B}$ induit un morphisme d'espaces k-affinoïdes $\varphi : Y \to X$ pour lequel on a :

$$\varphi(Y) = S. \tag{2}$$

Plus précisément, on a la trichotomie suivante. Si $x \in X$ et en notant Y_x la fibre de φ en x

$$\begin{cases} Y_x \simeq \mathcal{M}(\mathcal{H}(x)) \text{ si } |f(x)| \leq |g(x)| \neq 0 \\ Y_x \simeq \mathcal{M}(\mathcal{H}(x)\{T\}) = \mathbb{B}_{\mathcal{H}(x)} \text{ si } f(x) = g(x) = 0 \\ Y_x = \emptyset \text{ sinon, i.e. si } x \notin S \end{cases}$$

De manière plus précise d'ailleurs, si on note

$$R := \{x \in X \mid |f(x)| \leq |g(x)| \neq 0\},$$

alors

$$\varphi^{-1}(R) = \{y \in Y \mid g(y) \neq 0\},$$

et R (resp. $\varphi^{-1}(R)$) est un domaine analytique de X (resp. de Y). De plus,

$$\varphi_{|\varphi^{-1}(R)} : \varphi^{-1}(R) \to R$$

est un isomorphisme d'espaces k-analytiques. Quoi qu'il en soit, l'égalité (2) implique déjà que toute partie semi-analytique de X s'obtient comme une combinaison booléenne d'ensembles de la forme $\varphi(Y)$ où φ est un morphisme d'espaces k-affinoïdes. Comme nous l'avons cependant déjà précisé, la réciproque n'est pas vraie : si $\varphi : Y \to X$ est un morphisme entre espaces k-affinoïdes, $\varphi(Y)$ n'est pas semi-analytique en général.

0.3.2 Remarque strict/pas strict

Suivant [Duc03], nous aurions pu définir une partie semi-analytique de X par des inégalités $|f| \bowtie \lambda|g|$ avec $\lambda \in \mathbb{R}_+$, ou même parler de parties Δ-semi-analytiques, avec Δ un sous-groupe de R_+^*, en autorisant $\lambda \in \Delta$. D'ailleurs, suivant la terminologie qu'il est classique d'utiliser dans les espaces de Berkovich, nous aurions du appeler les parties définies en 0.3.1 *strictement semi-analytiques*. Nous ne l'avons pas fait pour alléger la présentation. Le lecteur qui souhaiterait utiliser des parties semi-analytiques définies par des inégalités $|f| \bowtie \lambda|g|$ avec $\lambda \in \mathbb{R}_+$ peut le faire, et tous les résultats de cette thèse se transposent dans ce contexte. Cela découle du fait qu'en dernier ressort, nos résultats reposent sur un théorème d'élimination des quantificateurs algébrique tel que [Duc03, 2.5] qui autorise justement des constantes $\lambda \in \mathbb{R}$. Nous détaillons cela dans la section 1.1.6.

0.3.3 Variantes locales

Définition 0.3.6. *Une partie $S \subset X$ est*

– ***semi-analytique rigide*** *s'il existe $\{X_i\}$ un recouvrement affinoïde fini de X tel que pour tout i, $S \cap X_i$ est semi-analytique dans X_i.*

– ***localement semi-analytique*** *si pour tout $x \in X$ il existe U un voisinage affinoïde de x tel que $S \cap U$ est semi-analytique dans U.*

Pour une partie S de X on a les implications suivantes :

S est semi-analytique $\Rightarrow$ S est localement semi-analytique $\Rightarrow$ S est semi-analytique rigide.

La première implication est claire. La seconde découle du fait que pour tout point $x \in X$, les voisinages affinoïdes de x forment une base de voisinage de x pour la topologie de Berkovich sur X [Ber90, 2.2.3(iii)] et de la compacité de X.

Bien que nous ne l'utiliserons pas, mentionnons une autre variante :

Définition 0.3.7. *Une partie S de X est semi-analytique bancale s'il existe $\{X_i\}$ une famille finie de domaines affinoïdes de X et pour tout i, $S \subset X_i$ une partie semi-analytique de X_i telle que $S = \cup_i S_i$.*

On vérifie que cela équivaut aussi à dire que S est une combinaison booléenne de parties semi-analytiques de domaines affinoïdes de X (c'est la définition de [LR05, 6.1]). On a clairement l'implication

$$\text{semi-analytique rigide} \Rightarrow \text{semi-analytique bancal}$$

et cette implication est stricte en général (cf. [LR05, 6.2.(ii)]).

0.3.4 Ensembles semi-algébriques

Définition 0.3.8. *[Duc03, 2.1] Soit $\mathcal{X} = Spec(B)$ où B est une $\mathcal{A}$-algèbre de type fini. Une partie $S \subset \mathcal{X}^{an}$ est dite* ***semi-algébrique*** *si c'est une combinaison booléenne d'ensembles de la forme*

$$\{x \in \mathcal{X}^{an} \mid |f(x)| \leq |g(x)|\}$$

où $f, g \in B$.

Remarque 0.3.9. *Précisons que cette définition dépend de la donnée de $\mathcal{X}$ et pas seulement de l'espace k-analytique $\mathcal{X}^{an}$.*

Par exemple, soit $\mathcal{A} = k$ et $\mathcal{X} = Spec(k[T_1, T_2])$, de sorte que

$$\mathcal{X}^{an} \simeq \mathbb{A}_k^{2,an}. \tag{3}$$

Considérons $S = \{x \in \mathbb{A}_k^{2,an} \mid T_2(x) = 0\}$. Alors, S est semi-algébrique dans $\mathbb{A}_k^{2,an}$ par rapport à la présentation (3).

Soit maintenant $f = \sum_{n \geq 0} a_n T_1^n$ une série avec $a_n \in k$ dont le rayon de convergence est infini, et supposons que f n'est pas un polynôme. On considère alors l'automorphisme de $\mathbb{A}_k^{2,an}$ défini par

$$\begin{array}{rccc} \varphi : & \mathbb{A}_k^{2,an} & \rightarrow & \mathbb{A}_k^{2,an} \\ & (T_1, T_2) & \mapsto & (T_1, T_2 + f(T_1)) \end{array}$$

Il est facile de voir que $\varphi(S) \subset \mathbb{A}_k^{2,an}$ n'est pas semi-algébrique par rapport à la présentation (3). En effet, $\varphi(S)$ est défini par l'équation $T_2 = f(T_1)$. Si cette partie était semi-algébrique, la partie définie par $T_2 = f(T_1)$ et $T_2 = 0$ devrait aussi être semi-algébrique, cependant elle est en bijection avec les zéros de f. Or une telle série entière qui a un rayon de convergence *infini et qui n'est pas un polynôme a une infinité dénombrable de zéros*[4]*. Et une partie semi-algébrique (de* $(\mathbb{A}_k^1)^{an}$ *en l'occurrence ici) ne peut être infinie dénombrable.*

Ainsi, pour être précis, il faudrait dire que S *est une partie semi-algébrique de* $\mathcal{X}^{an}$ *par rapport à* $\mathcal{X}$*. Cependant, nous tairons souvent ce détail, la présentation algébrique sous-jacente étant claire suivant le contexte.*

Lemme 0.3.10. *Soit* $\mathcal{A}$ *une algèbre k-affinoïde,* $\mathcal{X}$ *un* $\mathcal{A}$*-schéma affine de type fini et soit* $\mathcal{X} = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n$ *un recouvrement affine de* $\mathcal{X}$*. Si* $S \subset \mathcal{X}^{an}$*, alors* S *est semi-algébrique dans* $\mathcal{X}^{an}$ *si et seulement si* $S \cap \mathcal{U}_i^{an}$ *est semi-algébrique dans* $\mathcal{U}_i^{an}$ *pour tout* i*.*

Démonstration. Soit $S \subset \mathcal{X}^{an}$.

D'une part, il suit de la définition 0.3.8 que si S est semi-algébrique dans $\mathcal{X}^{an}$, alors $S \cap \mathcal{U}_i^{an}$ est aussi semi-algébrique dans $\mathcal{U}_i^{an}$.

Réciproquement, si $S \cap \mathcal{U}_i^{an}$ est semi-algébrique[5] dans $\mathcal{U}_i^{an}$ pour tout i, alors d'après le théorème d'élimination des quantificateurs 0.4.7, $S \cap \mathcal{U}_i^{an}$ est aussi semi-algébrique dans $\mathcal{X}^{an}$. Ainsi, $S = \cup_i S \cap \mathcal{U}_i^{an}$ est semi-algébrique dans $\mathcal{X}^{an}$. □

Lemme 0.3.11. *Soit* $\mathcal{X}$ *un* $\mathcal{A}$*-schéma de type fini, et* $S \subset \mathcal{X}^{an}$*. Soit* $\mathcal{X} = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n$ *un recouvrement affine de* $\mathcal{X}$*. On a les équivalences suivantes :*

1. *Pour* $i = 1 \ldots n$*,* $S \cap \mathcal{U}_i^{an}$ *est semi-algébrique dans* $\mathcal{U}_i^{an}$*.*
2. *Pour tout ouvert affine* $\mathcal{V} \subset \mathcal{X}$*,* $S \cap \mathcal{V}^{an}$ *est semi-algébrique dans* $\mathcal{V}^{an}$*.*

Démonstration. Supposons la condition (1) vérifiée. Soit $\mathcal{V}$ un ouvert affine de $\mathcal{X}$. Alors $\mathcal{U}_i \cap \mathcal{V}$ est réunion finie d'ouverts affines $\mathcal{V}_{i,j}$ de $\mathcal{V}$, et puisque $S \cap \mathcal{U}_i^{an}$ est semi-algébrique dans $\mathcal{U}_i^{an}$, $S \cap (\mathcal{V}_{i,j})^{an}$ est semi-algébrique dans $(\mathcal{V}_{i,j})^{an}$. Puisque la famille $\{\mathcal{V}_{i,j}\}_{i,j}$ est un recouvrement fini de $\mathcal{V}$ et comme $S \cap \mathcal{V}^{an} = \cup_{i,j} S \cap (\mathcal{V}_{i,j})^{an}$, d'après le lemme précédent, $S \cap \mathcal{V}^{an}$ est semi-algébrique dans $\mathcal{V}^{an}$, i.e. (2) est vérifiée.

Réciproquement, si l'on suppose (2), pour montrer (1) il suffit d'appliquer (2) avec $\mathcal{V} = \mathcal{U}_i$. □

Définition 0.3.12. *Soit* $\mathcal{X}$ *un* $\mathcal{A}$*-schéma séparé de type fini. Une partie* $S \subset \mathcal{X}^{an}$ *est dite* ***semi-algébrique*** *si elle satisfait l'une des conditions équivalentes du lemme 0.3.11.*

Remarque 0.3.13. *D'après 0.3.11, on peut vérifier si* $S \subset \mathcal{X}^{an}$ *est semi-algébrique en prenant un recouvrement affine de* $\mathcal{X}$*, et cela ne dépend pas du recouvrement. En particulier, si* $\mathcal{X}$ *est affine, le lemme 0.3.10, permet d'affirmer que les deux définitions 0.3.8 et 0.3.12 d'une partie semi-algébrique sont équivalentes.*

De même on prouve sans difficulté la généralisation suivante du théorème de projection pour les semi-algébriques.

Proposition 0.3.14. *Soit* $\mathcal{X}$ *et* $\mathcal{Y}$ *deux* $\mathcal{A}$*-schémas séparés de type fini,* $f : \mathcal{X} \to \mathcal{Y}$ *un* $\mathcal{A}$*-morphisme, et* S *une partie semi-algébrique de* $\mathcal{X}^{an}$*. Alors* $f^{an}(S)$ *est une partie semi-algébrique de* $\mathcal{Y}^{an}$*.*

Remarque 0.3.15. *Nous voulons insister sur le fait que la définition 0.3.12 généralise à la fois la définition d'une partie semi-analytique de* $X = \mathcal{M}(\mathcal{A})$ *et celle, plus classique des parties semi-algébriques de* $\mathcal{X}^{an}$ *où* $\mathcal{X}$ *est un k-schéma affine de type fini.*

---

4. Cela découle du fait qu'une telle fonction a forcément un zéro. Pour s'en convaincre, considérons $\lambda \in k^*$ tel que pour un $n \geq 1$, on ait $|a_n \lambda^n| \geq |a_0|$. Alors si l'on considère $f(\lambda T)$ comme un élément de $k\{T\}$, il est forcément distingué d'ordre s avec $s \geq 1$. D'après le théorème de préparation de Weierstrass, on en déduit que f admet s racines dans le disque de rayon λ.

5. Quand $\mathcal{U}$ est un ouvert affine principal de la forme $D(s)$, cela se voit facilement.

Remarque 0.3.16. *Soit $\mathcal{X}$ un $\mathcal{A}$-schéma affine de type fini, V un domaine affinoïde de $\mathcal{X}^{an}$, et soit $S \subset \mathcal{X}^{an}$ une partie semi-algébrique de $\mathcal{X}^{an}$. Alors $S \cap V$ est semi-analytique dans V.*

Lemme 0.3.17. *Soit $\mathcal{X}$ un $\mathcal{A}$-schéma de type fini, séparé, $S \subset \mathcal{X}^{an}$ une partie semi-algébrique de $\mathcal{X}^{an}$ et V un domaine affinoïde de $\mathcal{X}^{an}$. Alors $S \cap V$ est semi-analytique rigide dans V.*

Démonstration. Il est possible de trouver un recouvrement fini de V par des domaines affinoïdes $V = \cup_{i=1}^{n} V_i$, et pour tout i un ouvert affine $\mathcal{U}_i$ de $\mathcal{X}$ tel que pour tout i $V_i \subset \mathcal{U}_i^{an}$. Alors $S \cap V_i = (S \cap \mathcal{U}_i^{an}) \cap V_i$, et puisque $S \cap \mathcal{U}_i^{an}$ est semi-algébrique dans $\mathcal{U}_i^{an}$ et $V_i \subset \mathcal{U}_i^{an}$, $S \cap V_i$ est semi-analytique dans V_i d'après la remarque 0.3.16. □

En fait, $S \cap V$ est même localement semi-analytique. Nous ne savons pas s'il est même semi-analytique.

0.3.5 Le cas particulier de la dimension 1

On présente rapidement la situation particulièrement simple du disque unité fermé $X = \mathbb{B}$.

Lemme 0.3.18. *Une partie S de $\mathbb{B}$ est semi-analytique si et seulement si elle est définie par des inégalités polynomiales, i.e. de la forme $|f| \leq |g|$ avec $f, g \in k[T]$.*

Démonstration. D'après le théorème de préparation de Weierstrass avec $n = 1$, si $f \in k\{T\}$, il existe un polynôme $P \in k\{T\}$ et un élément inversible $e \in k\{T\}$ inversible tel que $f = eP$. Comme e est inversible, pour tout $x \in \mathbb{B}$, $|e(x)| = \|e\| \in \sqrt{|k^\times|}$, et le résultat en découle. □

Les parties sous-analytiques seront définies dans la section 0.5. C'est la seule bonne classe que l'on connaisse qui contienne les images de morphismes entre espaces affinoïdes. En règle général, elle contient strictement toutes les autres classes que nous mentionnons, en particulier celle des parties semi-analytiques. Sauf en dimension 1 :

Proposition 0.3.19 ([LR96, Corollaire 4.7])**.** *Dans $\mathbb{B}$ les parties semi-analytiques et sous-analytiques coïncident.*

0.4 Élimination des quantificateurs, théorie ACVF

On présente le théorème d'élimination des quantificateurs pour ACVF, et on explique en quoi ce théorème équivaut à un énoncé de stabilité par image des semi-algébriques, et comment des énoncés de théorie des modèles se traduisent en terme d'espaces k-analytiques. Signalons dans cet esprit [6] des résultats récents [HL10] de modération topologique d'analytifiés $\mathcal{X}^{\text{an}}$ de variétés quasi-projectives, dont on peut trouver une exposition dans [Duc12b].

6. Mais avec une utilisation d'outils bien plus sophistiqués que l'élimination des quantificateurs.

0.4.1 Théorie des modèles de ACVF

On appelle corps valué un corps k muni d'une application $|\cdot| : k \to \Gamma \cup \{0\}$ où $(\Gamma, <)$ est un groupe abélien totalement ordonné (que l'on note multiplicativement), vérifiant $|x| = 0 \Leftrightarrow x = 0$, $|x+y| \leq \max(|x|, |y|)$ et $|xy| = |x||y|$. Pour donner un sens à cela, on prolonge l'ordre $<$ à $\Gamma \cup \{0\}$ en posant $0 < \gamma$ pour tout $\gamma \in \Gamma$ et $0 \cdot \gamma = \gamma \cdot 0 = 0$. On suppose de plus que $|k^*|$ engendre le groupe Γ. On appelle Γ le groupe des valeurs de la valuation. Quand $(\Gamma, <)$ est isomorphe à un sous groupe ordonné de $(\mathbb{R}^*_+, <)$ on dit que la valuation est de rang 1 (cf [Bou98, VI.4] pour plus de détails), et cela revient à étudier les corps non archimédiens (modulo le choix d'un plongement de Γ dans $(\mathbb{R}^*_+, <)$).

Si k est un corps muni d'une valuation à valeur dans Γ et K un corps muni d'une valuation à valeur dans Δ, un morphisme de corps valués est la donnée d'un morphisme de corps $k \to K$ et d'un morphisme de groupe abélien ordonnés : $\Gamma \to \Delta$ (que l'on prolonge en $\Gamma \cup \{0\} \to \Delta \cup \{0\}$ en envoyant 0_Γ sur 0_Δ) rendant commutatif

$$\begin{array}{ccc} k & \xrightarrow{\alpha} & K \\ {\scriptstyle |\cdot|}\downarrow & & \downarrow{\scriptstyle |\cdot|} \\ \Gamma \cup \{0\} & \xrightarrow[\beta]{} & \Delta \cup \{0\} \end{array}$$

La flèche α est injective (car c'est un morphisme de corps), mais la flèche β ne l'est pas forcément. C'est le cas précisément quand on peut identifier k à un sous-corps valué de K et dans ce cas, on dit que K est une **extension valuée** de k.

On fixe un corps valué $(k, |\cdot|)$. On appelle formule à paramètres dans k une formule obtenue avec des scalaires $\lambda \in k$, les opérateurs du corps $+, -, \cdot$ de k, des variables $x_1, \ldots, y_1, \ldots$, la fonction $|\cdot|$ et les symboles $\cdot, <, \leq$ sensés opérer sur le groupe ordonné, le symbole $=$, les connecteurs logiques $\wedge, \vee, \neg$ et le droit de quantifier des variables, i.e. d'écrire $\forall x, \ldots$ et $\exists y, \ldots$. Si on fixe une constante $\kappa \in k$, voici trois formules.

$$\begin{array}{lr} \varphi_1 & \exists y, y^2 = x \\ \varphi_2 & \exists y, \Big((y^2 = x(x-1)(x+1)) \wedge (|x| \leq 1)\Big) \\ \varphi_3 & x + y = \kappa \end{array}$$

On écrira $\varphi(x_1, \ldots, x_n)$ pour exprimer que toutes les variables qui apparaissent dans la formule φ sont dans $\{x_1, \ldots, x_n\}$. Une formule est dite sans quantificateurs si elle ne fait pas intervenir de quantificateurs $\exists$ et $\forall$. Par exemple, seule φ_3 est sans quantificateurs. Une variable x est dite libre dans la formule φ si elle ne dépend pas d'un quantificateur dans φ. Par exemple dans φ_1 et φ_2, x est libre, mais pas y et dans φ_3, x et y sont libres.

Soit $\varphi(x_1, \ldots, x_n)$ une formule à paramètres dans k, K une extension valuée de k, et $(a_1, \ldots, a_n) \in K^n$, on notera

$$K \models \varphi(a_1, \ldots, a_n)$$

si la formule φ où on a remplacé les x_i par a_i est vraie. Par exemple si $k = \mathbb{Q}_p$,

$$\mathbb{Q}_p \not\models \varphi_1(p)$$

où $\varphi_1(p)$ est la formule $\exists y, y^2 = p$. En revanche

$$\mathbb{C}_p \models \varphi_1(p).$$

Théorème 0.4.1 (Élimination des quantificateurs dans ACVF)**.** *Soit $\varphi(x_1, \ldots, x_n)$ une formule à paramètres dans k. Alors il existe une formule à paramètres dans k sans quantificateurs $\psi(x_1, \ldots, x_n)$ telle que pour toute extension non trivialement valuée $k \to K$ avec K algébriquement clos, et pour tout n-uplet $(a_1, \ldots, a_n) \in K^n$,*

$$K \models \varphi(a_1, \ldots, a_n) \quad \textit{si et seulement si} \\ K \models \psi(a_1, \ldots, a_n)$$

Dit autrement, du point de vue de K les formules φ et ψ sont équivalentes. Il est fondamental que dans ce théorème, la formule ψ **marche pour toute extension non trivialement valuée algébriquement close** K. Dit autrement, cette procédure est uniforme en K.

Ce résultat est d'habitude formulé de la manière suivante. On considère d'abord $\mathcal{L}_{div} = \{+, -, \cdot, 0, 1, |\}$ le langage des anneaux avec une valuation, c'est à dire que $+, -, \cdot$ sont opérations binaires, $0, 1$ des constantes et $|$ un symbole de relation. La théorie des corps non trivialement valués algébriquement clos, T_{ACVF} est l'ensemble des énoncés du premier ordre dans le langage $\mathcal{L}_{div}$ constitués :

– des formules définissant un corps. Par exemple

$$\forall x, (x \neq 0) \to (\exists y, xy = 1).$$

Il n'y a qu'un nombre fini de telles formules.

– des formules faisant qu'on a un corps non trivialement valué, où la valuation est défini par $|x| \leq |y|$ si et seulement si $x|y$. Il n'y en a également qu'un nombre fini, par exemple :

$$\forall x, y, z, (x|y \wedge x|z) \to x|(y + z) \\ \exists x, (x \neq 0) \wedge (\neg(x|1)).$$

– des formules disant qu'on a un corps algébriquement clos. Il y a une infinité de formules, les φ_n pour $n \geq 1$:

$$\varphi_n = \ \forall a_0, a_1, \ldots, a_{n-1}, \exists x, x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0.$$

Les modèles des T_{ACVF} sont alors exactement les corps non trivialement valué algébriquement clos. Le théorème d'élimination des quantificateurs est alors

Théorème 0.4.2. *La théorie T_{ACVF} élimine les quantificateurs. Dit autrement, pour toute formule φ dans le langage $\mathcal{L}_{div}$, il existe une formule sans quantificateur ψ telle que*

$$T_{ACVF} \models \varphi \leftrightarrow \psi. \tag{4}$$

La formule (4) signifie que pour tout modèle K de ACVF, i.e. pour tout corps non trivialement valué algébriquement clos K, $K \models \varphi$ si et seulement si $K \models \psi$. En vertu du théorème de complétude de Gödel, cela signifie aussi que l'équivalence $\varphi \leftrightarrow \psi$ peut être formellement déduite des axiomes de T_{ACVF}. Une référence que l'on trouve habituellement pour ce théorème est [Rob77]. Le lecteur francophone pourra consulter les notes de cours [Cha08, chapitre 2]

Il nous faut conclure avec ces remarques tautologiques, mais qui font le lien avec des aspects plus géométriques :

Fait 0.4.3. *Soit S une partie de $(\mathbb{A}^n_k)^{rig}$. On a équivalence entre*

– *S est semi-algébrique dans $(\mathbb{A}_k^n)^{rig}$*
– *Il existe une formule sans quantificateurs $\varphi(x_1, \dots, x_n)$ à paramètres dans k telle que, modulo l'identification de $(\mathbb{A}_k^n)^{rig}$ avec les orbites sous Galois de $(k^{alg})^n$*

$$S = \{(a_1, \dots, a_n) \in (k^{alg})^n \mid k^{alg} \models \varphi(a_1, \dots, a_n)\}.$$

De même, si S est une partie de $(\mathbb{A}_k^n)^{an}$, on a équivalence entre
– *S est semi-algébrique dans $(\mathbb{A}_k^n)^{an}$.*
– *Il existe une formule sans quantificateurs $\varphi(x_1, \dots, x_n)$ à paramètres dans k telle que*

$$S = \{x \in (\mathbb{A}_k^n)^{an} \mid \mathcal{H}(x) \models \varphi(x_1, \dots, x_n)\}$$

où dans la formule ci-dessus, on a noté pour $x \in (\mathbb{A}_k^n)^{an}$, x_i l'image de X_i dans $\mathcal{H}(x)$ par le morphisme d'évaluation $k[X_1, \dots, X_n] \to \mathcal{H}(x)$.

0.4.2 De la théorie des modèles à des énoncés géométriques

On va détailler comment le théorème d'élimination des quantificateurs dans ACVF se traduit dans le langage des espaces rigides puis des espaces de Berkovich.

Proposition 0.4.4 (Projection des semi-algébriques, version 1)**.** *Soit $\pi : (\mathbb{A}_k^{n+m})^{rig} \to (\mathbb{A}_k^n)^{rig}$ la première projection et S une partie semi-algébrique de $(\mathbb{A}_k^{n+m})^{rig}$. Alors $\pi(S)$ est une partie semi-algébrique de $(\mathbb{A}_k^n)^{rig}$.*

Nous allons en donner une preuve, mais voulons insister sur le fait que c'est un corollaire immédiat du théorème d'élimination des quantificateurs dans $ACVF$.

Démonstration. D'après la remarque 0.3.4 on peut supposer que S est semi-algébrique basique, i.e. de la forme

$$S = \bigcap_{i=1}^{N} \{z \in (\mathbb{A}_k^{n+m})^{\mathrm{rig}} \mid |f_i(z)| \bowtie_i |g_i(z)|\}$$

avec $f_i, g_i \in k[X_1, \dots, X_n, Y_1, \dots, Y_m]$. On lui associe alors la formule à paramètres dans k

$$\varphi(x_1, \dots, x_n) = \Big(\exists y_1, \dots, y_m, \bigwedge_{i=1}^{N} |f_i(x_1, \dots, x_n, y_1, \dots, y_m)| \bowtie_i |g_i(x_1, \dots, x_n, y_1, \dots, y_m)|\Big).$$

D'après le théorème d'élimination des quantificateurs dans ACVF, il existe une formule à paramètres dans k sans quantificateurs $\psi(x_1, \dots, x_n)$ qui soit équivalente dans k^{alg} à $\varphi(x_1, \dots, x_n)$. Or $(\mathbb{A}_k^{n+m})^{\mathrm{rig}}$ (resp. $(\mathbb{A}_k^n)^{\mathrm{rig}}$) s'identifie avec $(k^{\mathrm{alg}})^{n+m}$ (resp. $(k^{\mathrm{alg}})^n$) modulo l'action de Galois, et k^{alg} est une extension de corps valués k, et modulo ces identifications,

$$\pi(S) = \Big\{(x_1, \dots, x_n) \in k^{\mathrm{alg}} \mid \exists y_1, \dots, y_m \in (k^{\mathrm{alg}})^m \Big(\bigwedge_{i=1}^{N} |f_i(x_1, \dots, x_n, y_1, \dots, y_m)| \bowtie_i |g_i(x_1, \dots, x_n, y_1, \dots, y_m)|\Big)\Big\}$$

et donc, toujours modulo l'action de Galois, on obtient

$$\pi(S) = \Big\{(x_1, \dots, x_n) \in k^{\mathrm{alg}} \mid k^{\mathrm{alg}} \models \psi(x_1, \dots, x_n)\Big\}$$

qui est bien semi-algébrique (cf fait 0.4.3). □

Proposition 0.4.5 (Projection des semi-algébriques, version 2)**.** *Soit* $\pi : (\mathbb{A}_k^{n+m})^{an} \to (\mathbb{A}_k^n)^{an}$ *la première projection et* S *une partie semi-algébrique de* $(\mathbb{A}_k^{n+m})^{an}$*. Alors* $\pi(S)$ *est une partie semi-algébrique de* $(\mathbb{A}_k^n)^{an}$*.*

Démonstration. La preuve est très proche, mais il faut rajouter une subtilité. On considère donc [7]

$$S = \{z \in (\mathbb{A}_k^{n+m})^{\mathrm{an}} \mid |f(z)| \bowtie |g(z)|\}$$

avec $f, g \in k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$. On lui associe alors la formule à paramètres dans k

$$\varphi(x_1, \ldots, x_n) = \Big(\exists y_1, \ldots, y_m, |f(x_1, \ldots, x_n, y_1, \ldots, y_m)| \bowtie |g(x_1, \ldots, x_n, y_1, \ldots, y_m)|\Big).$$

D'après le théorème d'élimination des quantificateurs dans ACVF, il existe une formule à paramètres dans k sans quantificateurs $\psi(x_1, \ldots, x_n)$ qui soit équivalente à $\varphi(x_1, \ldots, x_n)$ dans toute extension valuée algébriquement close K. À cette formule sans quantificateurs ψ, on associe (fait 0.4.3) une partie semi-algébrique T de $(\mathbb{A}_k^n)^{\mathrm{an}}$.

Soit maintenant $x \in (\mathbb{A}_k^n)^{\mathrm{an}}$. D'une part, $x \in \pi(S) \Leftrightarrow \pi^{-1}(x) \cap S \neq \emptyset$. D'autre part $\pi^{-1}(x)$ s'identifie à $(\mathbb{A}_{\mathcal{H}(x)}^m)^{\mathrm{an}}$, et via cette identification, $\pi^{-1}(x) \cap T$ s'identifie à

$$S_x := \{z \in (\mathbb{A}_{\mathcal{H}(x)}^m)^{\mathrm{an}} \mid |f_x(z)| \bowtie g_x(z)|\}$$

où on a posé

$$f_x = f(x_1, \ldots, x_m, Y_1, \ldots, Y_m) \in \mathcal{H}(x)[Y_1, \ldots, Y_n]$$

$x_i \in \mathcal{H}(x)$ correspondant à l'image de X_i par le morphisme d'évaluation $k[X_1, \ldots, X_n] \to \mathcal{H}(x)$ (et de même pour g_x). On est ramené à savoir si $S_x \subset (\mathbb{A}_{\mathcal{H}(x)}^m)^{\mathrm{an}}$ est vide. En utilisant l'équivalence entre les semi-normes et les caractères, cela équivaut s'il existe une extension non archimédienne K de $\mathcal{H}(x)$ telle que S_x est un point à coordonnées dans K, et on peut, quitte à prendre sa clôture algébrique, supposer que K est algébriquement clos. On a alors

$x \in \pi(S) \Leftrightarrow S_x \neq \emptyset$

$\Leftrightarrow$ il existe un extension non archimédienne algébriquement close K de $\mathcal{H}(x)$ telle que $\exists (y_1, \ldots, y_m) \in K^m, |f(x_1, \ldots, x_n, y_1, \ldots, ym)| \bowtie |g(x_1, \ldots, x_n, y_1, \ldots, y_m)|$

$\Leftrightarrow$ il existe un extension non archimédienne algébriquement close K de $\mathcal{H}(x)$ telle que (Unif)

$K \models \psi(x_1, \ldots, x_n)$

$\Leftrightarrow x \in T.$

Ici l'équivalence à la ligne (Unif) vient de l'uniformité dans l'élimination des quantificateurs : ψ marche indépendamment de K. Finalement, $\pi(S) = T$ est semi-algébrique. □

Proposition 0.4.6 (Projection des semi-algébriques, version 3)**.** *Soit* $\mathcal{A}$ *une algèbre* k*-affinoïde,* $X = \mathcal{M}(\mathcal{A})$ *l'espace* k*-affinoïde associé et* $\pi : X \times (\mathbb{A}_k^{n+m})^{an} \to X \times (\mathbb{A}_k^n)^{an}$ *la première projection. Soit* S *une partie semi-algébrique de* $X \times (\mathbb{A}_k^{n+m})^{an}$*. Alors* $\pi(S)$ *est une partie semi-algébrique de* $X \times (\mathbb{A}_k^n)^{an}$*.*

Démonstration. La preuve est encore similaire, mais il faut rajouter encore une subtilité. On considère

$$S = \{z \in (X \times \mathbb{A}_k^{n+m})^{\mathrm{an}} \mid |f(z)| \bowtie |g(z)|\}$$

7. On devrait plutôt considérer une intersection finie, mais on ne l'a pas fait pour alléger les notations.

avec $f, g \in \mathcal{A}[T_1, \ldots, T_n, Y_1, \ldots, Y_m]$. La difficulté pour appliquer le théorème d'élimination des quantificateurs est que f et g ne sont pas des formules à paramètres dans k, mais dans $\mathcal{A}$. Ce qu'on va faire, c'est qu'on va les considérer comme des formules à paramètres dans les $\mathcal{H}(x)$ pour x variant dans $X \times (\mathbb{A}_k^n)^{\mathrm{an}}$.

Par définition, il existe I une partie finie de $\mathbb{N}^m$ telle que

$$f = \sum_{\nu \in I} a_\nu Y^\nu$$
$$g = \sum_{\mu \in I} b_\mu Y^\mu$$

où pour $\nu, \mu \in I$, a_ν et $b_\nu \in \mathcal{A}[T_1, \ldots, T_n]$.

Introduisons alors deux ensembles de variables $\{A_\nu\}_{\nu \in I}$ et $\{B_\mu\}_{\mu \in I}$. D'après le théorème d'élimination des quantificateurs, il existe une formule sans quantificateurs $\varphi(A_\nu, B_\mu)_{\nu,\mu \in I}$ à paramètres dans k telle que pour toute extension valuée algébriquement close $k \to K$ on ait l'équivalence dans K

$$\exists y_1, \ldots, y_m \in K^m,\ |\sum_{\nu \in I} A_\nu y^\nu| \bowtie |\sum_{\mu \in I} B_\mu y^\mu| \Leftrightarrow \varphi(A_\nu, B_\mu).$$

On associe alors à $\varphi(A_\nu, B_\mu)$ une formule à paramètres dans k sans quantificateurs $\psi(A_\nu, B_\mu)$ équivalente. On introduit alors la partie semi-algébrique R de $X \times (\mathbb{A}_k^n)^{\mathrm{an}}$ défini par la formule $\psi(a_\nu, b_\mu)$. On peut le faire car a_ν et $b_\mu \in \mathcal{A}[T_1, \ldots, T_n]$.

Soit alors $x \in X \times (\mathbb{A}_k^n)^{\mathrm{an}}$. Il lui est associé un morphisme d'évaluation que l'on note

$$\begin{array}{ccc} \mathcal{A}[T_1, \ldots, T_n] & \to & \mathcal{H}(x) \\ a & \mapsto & a(x) \end{array}$$

Notons alors

$$f_x = \sum_{\nu \in I} a_\nu(x) Y^\nu$$
$$g_x = \sum_{\mu \in I} a_\mu(x) Y^\nu$$

Alors si l'on note

$$S_x = \{z \in (\mathbb{A}^m_{\mathcal{H}(x)})^{\mathrm{an}} \mid |f_x(z)| \bowtie |g_x(z)|\}$$

on a les équivalences

$$\begin{aligned} x \in \pi(T) &\Leftrightarrow S_x \neq \emptyset \\ &\Leftrightarrow \text{il existe un extension non archimédienne algébriquement close } K \text{ de } \mathcal{H}(x) \text{ telle que} \\ &\exists (y_1, \ldots, y_m) \in K^m\ |\sum_{\nu \in I} a_\nu(x) y^\nu| \bowtie |\sum_{\mu \in I} b_\mu y^\mu| \\ &\Leftrightarrow \text{il existe un extension non archimédienne algébriquement close } K \text{ de } \mathcal{H}(x) \text{ telle que} \\ &K \models \psi(a_\nu(x), b_\mu(x)) \\ &\Leftrightarrow x \in R \end{aligned}$$

□

Théorème 0.4.7 (Projection des semi-algébriques, version 4, [Duc03])**.** *Soit $\mathcal{A}$ une algèbre k-affinoïde, notons $\mathcal{X} = Spec(A)$, soit $\mathcal{Y}, \mathcal{Z}$ deux $\mathcal{X}$-schémas affines de type finis, $f : \mathcal{Y} \to \mathcal{Z}$ un morphisme de $\mathcal{A}$-schémas et S une partie semi-algébrique de $\mathcal{Y}^{an}$. Alors $f^{an}(S)$ est une partie semi-algébrique de $\mathcal{Z}^{an}$.*

Démonstration. On considère d'abord $\mathcal{Y}$ (resp. $\mathcal{Z}$) comme un sous-schéma fermé de $\mathcal{X} \times \mathbb{A}^n$ (resp. $\mathcal{X} \times \mathbb{A}^m$). Le graphe de f s'identifie alors à un sous-schéma fermé de $\mathcal{X} \times \mathbb{A}^{m+n}$. Cela permet de se ramener au cas de la Version 3 ci-dessus. □

0.5 Ensembles sous-analytiques

0.5.1 Propriétés fondamentales

Pour rester fidèle aux notations utilisées dans les articles [Lip93,LR00d], nous noterons K notre corps non archimédien.

Les ensembles sous-analytiques que l'on va considérer tirent leur origine de [Lip88], puis ont été plus précisément définis dans [Lip93] par L. Lipshitz. Ensuite, une étude plus systématique a été faite dans [LR00d] et une série d'articles [LR96, LR98, LR99, LR00a]. Rappelons que Γ dénote le groupe des valeurs de K^{alg}, c'est donc un groupe abélien totalement ordonné et totalement divisible.

Les ensembles semi-analytiques n'étant pas suffisants pour décrire les images de morphismes entre espaces affinoïdes, il faut trouver une autre idée. Celle développée par Lipshitz dans [Lip93] (et dont on trouve un analogue pour les sous-ensembles de $\mathbb{Q}_p^n$ dans [DvdD88]) est d'introduire un ensemble de fonctions plus gros que $k\langle X_1, \dots, X_n\rangle$. La définition précise qui va venir repose sur deux idées. La première consiste à faire des divisions. Grossièrement, si f et g sont deux fonctions, on veut considérer quelque chose comme $\frac{f}{g}$. La seconde consiste à introduire des anneaux de fonctions analytiques sur les boules ouvertes. Plus précisément, pour $m, n \in \mathbb{N}$, Lipshitz introduit un anneau [8] de fonctions analytiques sur $\left((K^{\text{alg}})^\circ\right)^m \times \left((K^{\text{alg}})^{\circ\circ}\right)^n$ qui est noté $S_{m,n}$. Ces anneaux $S_{m,n}$ sont étudiés en détail dans [LR00d] et, à part pour le lemme de normalisation de Noether, ils vérifient toutes les bonnes propriétés de $k\langle X_1, \dots, X_n\rangle$, en particulier sont noethériens. Nous renvoyons à [LR00d, p.9] où une liste de ces propriétés est donnée.

Les anneaux $S_{m,n}$ dépendent en fait du choix (cf [LR00d, 2.1.1]) d'un sous-anneau de valuation discrète E de K° et devraient donc plus rigoureusement être notés $S_{m,n}(E, K)$, mais on les notera $S_{m,n}$ quand E sera clair dans le contexte.

Soit $\{a_i\}_{i\in\mathbb{N}}$ une suite d'élément de K° qui tend vers 0. On lui associe le sous-anneau de K° :

$$\left(E[a_0, a_1, \dots]_{\{a\in E[a_0,a_1,\dots]\ |\ |a|=1\}}\right)^{\widehat{}}.$$

Notons $\mathfrak{B}$ l'ensemble des sous-anneaux de K° de cette forme.

Définition 0.5.1 ([LR00d, 2.1.1]).

$$S_{m,n}(E, K) := K \otimes_{K^\circ} \Big(\varinjlim_{B\in\mathfrak{B}} \left(B\langle X_1, \dots, X_m\rangle[[\rho_1, \dots, \rho_n]]\right)\Big).$$

On va donner des descriptions plus concrètes dans certains cas.

Quand $\widetilde{K}$ est algébrique sur $\widetilde{E}$, on a [LR00d, 2.1.3 (i)] :

$$S_{m,n}(E, K) = K \widehat{\otimes}_E \left(E\langle X_1 \dots X_m\rangle[[\rho_1 \dots \rho_n]]\right). \tag{5}$$

Ainsi, quand E vérifie l'hypothèse "$\widetilde{K}$ est algébrique sur $\widetilde{E}$", on prendra le membre droit de cette formule comme définition.

8. Dans [Lip93] ces anneaux sont notés $K\langle X\rangle[[\rho]]_s$ où la lettre s veut dire *séparé*.

Quand la valuation sur K est discrète, il est donc très naturel de prendre $E = K^\circ$. Dans ces conditions, la formule (5) devient

$$S_{m,n}(K^\circ, K) = K\widehat{\otimes}_{K^\circ}\Big(K^\circ\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_n]]\Big) = K\otimes_{K^\circ}\Big(K^\circ\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_n]]\Big).$$

Le cas géométrique

De notre point de vue, il est naturel de détailler le cas *géométrique*, i.e. le cas où K est algébriquement clos.

Dans [Lip93, 2.1], K est algébriquement clos, et les anneaux $S_{m,n}$ (qui y sont notés $K\langle x\rangle[[\rho]]_s$) dépendent des choix suivants. On fixe un réel $0 < r < 1$ avec $r \in |K^*|$, et $R_\circ \subset K^\circ$, un sous-anneau de valuation discrète de K° tel que $|R_\circ^*| = r^{\mathbb{N}}$ et $\widetilde{R_\circ} = \widetilde{K^\circ}$, et qui soit maximal pour ces propriétés. Un tel $R_\circ$ existe toujours, mais n'est pas nécessairement unique en général [9]. Quoi qu'il en soit, on se fixe un tel $R_\circ$ et, pour revenir à nos notations initiales, on pose $E := R_\circ$. Alors, d'après (5) on a

$$S_{m,n} = S_{m,n}(R_\circ, K) = K\widehat{\otimes}_{R_\circ}\big(R_\circ\langle X_1 \ldots X_m\rangle[[\rho_1 \ldots \rho_n]]\big).$$

Par exemple si $K = \mathbb{C}_p$ et que l'on choisit $r = |p|$, nous nous trouvons dans la situation où $R_\circ$ est unique, en l'occurrence, $R_\circ = \big(\widehat{\mathbb{Q}_p^{unr}}\big)^\circ = W((\mathbb{F}_p)^{alg})$. D'après (5) pour $K = \mathbb{C}_p$ on posera

$$S_{m,n} = \mathbb{C}_p\widehat{\otimes}_{\widehat{\mathbb{Q}_p^{unr}}^\circ}\Big((\widehat{\mathbb{Q}_p^{unr}}^\circ)\langle X_1 \ldots X_m\rangle[[\rho_1 \ldots \rho_n]]\Big).$$

Pour donner un autre exemple, si $K = \widehat{\cup_{n\geq 0}\mathbb{C}((t^{\frac{1}{n}}))}$ (qui est $\widehat{\mathbb{C}[[t]]^{alg}}$), alors si l'on choisit $r = |t|$, $\mathbb{C}[[t]]$ est l'unique $R_\circ$ qui convient et qui contient $\mathbb{C}[t]$. Dans ce cas, on posera

$$S_{m,n} = \widehat{\mathbb{C}[[t]]^{alg}}\widehat{\otimes}_{\mathbb{C}[[t]]}\Big((\mathbb{C}[[t]])\langle X_1 \ldots X_m\rangle[[\rho_1 \ldots \rho_n]]\Big).$$

En pratique, on on se fixe une bonne fois pour toute un E et on écrira $S_{m,n}$ à la place de $S_{m,n}(E, K)$.

Les D-fonctions

Ensuite, Lipshitz définit inductivement une K-algèbre de fonctions, l'algèbre des **D-functions** $f : \big((K^{\mathrm{alg}})^\circ\big)^n \to K^{alg}$ ainsi :

Définition 0.5.2.

1. *Si $f \in k\langle X_1, \ldots, X_n\rangle$, alors $f : \big((K^{alg})^\circ\big)^n \to K^{alg}$ est une D-fonction.*
2. *Si $f, g : \big((K^{alg})^\circ\big)^n \to K^{alg}$ sont des D-fonctions, alors $D_0(f, g)$ et $D_1(f, g)$ sont des D-fonctions $\big((K^{alg})^\circ\big)^n \to K^{alg}$ où,*

$$\begin{array}{rccc} D_0(f,g) : & \big((K^{alg})^\circ\big)^n & \to & K^{alg} \\ & x & \mapsto & \begin{cases} \frac{f(x)}{g(x)} & si\ |f(x)| \leq |g(x)| \neq 0 \\ 0 & sinon \end{cases} \end{array}$$

$$\begin{array}{rccc} D_1(f,g) : & \big((K^{alg})^\circ\big)^n & \to & K^{alg} \\ & x & \mapsto & \begin{cases} \frac{f(x)}{g(x)} & si\ |f(x)| < |g(x)| \\ 0 & sinon \end{cases} \end{array}$$

9. Nous voulons toutefois préciser que si $\widetilde{K}$ a un degré de transcendance fini sur son corps de base, alors $R_\circ$ est unique.

3. *Soient* $f_1, \dots, f_m, g_1, \dots g_n$ *des D-fonctions* $\left((K^{alg})^\circ\right)^p \to K^{alg}$. *Supposons que pour tout* $x \in \left((K^{alg})^\circ\right)^p$, *et pour tout* i, j *on ait* $|f_i(x)| \leq 1$ *et* $|g_j(x)| < 1$. *Et soit* $F \in S_{m,n}$. *Alors*

$$\begin{array}{rccc} F(f_1, \dots, f_m, g_1, \dots, g_n): & \left((K^{alg})^\circ\right)^p & \to & K^{alg} \\ & x & \mapsto & F(f_1(x), \dots, f_m(x), g_1(x), \dots, g_n(x)) \end{array}$$

est une D-fonction.

Définition 0.5.3. *Une partie* $S \subset \left((K^{alg})^\circ\right)^n$ *est* ***sous-analytique*** *si c'est une combinaison booléenne de parties de la forme* $\{x \in \left((K^{alg})^\circ\right)^n \mid |f(x)| \leq |g(x)|\}$ *où* f, g *sont des D-fonctions.*

Lipshitz prouve [Lip93, th. 3.8.2] que les parties sous-analytiques sont stables par projection. Ce théorème a la conséquence suivante : si $f : X \to Y$ est un morphisme d'espaces K-affinoïdes, et qu'on voit X (resp. Y) comme des fermés de Zariski de la boule $\left((K^{\text{alg}})^\circ\right)^p$ (resp. $\left((K^{\text{alg}})^\circ\right)^q$), alors $f(X)$ est une partie sous-analytique de $\left((K^{\text{alg}})^\circ\right)^q$.

Nous aurons besoin dans le chapitre 4 de la définition plus général suivante. Rappelons que nous notons Γ le groupe des valeurs du corps non archimédien K, défini par $\Gamma = \sqrt{|K^*|}$. C'est une sous-groupe de $\mathbb{R}_+^*$.

Définition 0.5.4. *Une partie* $S \subset \left((K^{alg})^\circ\right)^m \times \Gamma^n$ *est* ***sous-analytique*** *si c'est une combinaison booléenne de parties*

$$\left\{(x, \gamma) \in \left((K^{alg})^\circ\right)^m \times \Gamma^n \mid |f(x)|\gamma^u c \leq |g(x)|\right\}$$

où f, g *sont des D-fonctions,* $u \in \mathbb{Z}^n$ *et* $c \in \Gamma$. *Si* $n = 0$, *cela nous donne la définition que l'on avait déjà d'une partie sous-analytique* $S \subset \left((K^{alg})^\circ\right)^m$.
Si $m = 0$, *on obtient la notion d'ensemble définissable dans* $S \subset \Gamma^n$ *dans le langage des groupes abéliens totalement ordonnés (nous renvoyons à 4.1.4 pour plus de détails).*

Définition 0.5.5. *Une formule dans le langage* $\mathcal{L}_{an}^D$ *est une formule* $\varphi(x, \alpha)$ *du premier ordre en* x, γ, *i.e. écrite avec les symboles* $|f|$ *pour* f *une D-fonction, et les symboles* $\forall, \exists, |\cdot| \leq, <, =, +$.

Théorème 0.5.6. *[[Lip93, 3.8.2] et [LR00b, 4.2] pour l'aspect uniforme] Soit* $\varphi(x, \alpha)$ *une formule dans le langage* $\mathcal{L}_{an}^D$. *Alors il existe une formule sans quantificateurs* $\psi(x, \alpha)$ *telle que pour toute extension non archimédienne* F *de* K, *on ait*

$$(F^{alg})^\circ \models \varphi \leftrightarrow \psi.$$

Cet énoncé équivaut à

Théorème 0.5.7. *Soit* $S \subset \left((K^{alg})^\circ\right)^m \times \Gamma^n$ *un ensemble sous-analytique, et* $\pi : \left((K^{alg})^\circ\right)^m \times \Gamma^n \to \left((K^{alg})^\circ\right)^{m'} \times \Gamma^{n'}$ *une projection. Alors* $\pi(S)$ *est une partie sous-analytique de* $\left((K^{alg})^\circ\right)^{m'} \times \Gamma^{n'}$. *En particulier, si* $m' = 0$, $\pi(S)$ *est une partie de* $\Gamma^{n'}$ *définissable dans le langage des groupes abélien ordonnés.*

Remarque 0.5.8. *D'après nos définitions,* $S \subset \left((K^{alg})^\circ\right)^m \times \Gamma^n$ *est sous-analytique si et seulement si il existe une formule du premier ordre sans quantificateurs avec des D-fonctions telle que*

$$S = \{(x, \gamma) \in \left((K^{alg})^\circ\right)^m \times \Gamma^n \mid \varphi(x, \gamma) \text{ est vrai}\}.$$

Le théorème ci-dessus affirme que si l'on autorise φ à avoir des quantificateurs, S restera sous-analytique.

Si $\varphi(x,\gamma)$ est une formule du 1° ordre dans le langage $\mathcal{L}_{an}^D$, et $K \to L$ une extension non archimédienne, on posera

$$\varphi(L^{alg}) = \{(x,\gamma) \in ((L^{alg})^\circ)^m \times |L^{alg}|^n \mid \varphi(x,\gamma) \text{ est vrai}\}.$$

Remarque 0.5.9. *Une remarque sur l'uniformité. Si φ est une formule du première ordre sans quantificateurs dans le langage $\mathcal{L}_{an}^D$, alors elle définit une partie de $\left((K^{alg})^\circ\right)^m \times \Gamma^n$. Le théorème d'uniformité nous dit que la validité de φ ne sera pas modifiée, que l'on soit dans K^{alg} ou F^{alg}. En particulier, si $S \subset \left((K^{alg})^\circ\right)^m \times \Gamma^n$ est une partie sous-analytique définie par la formule φ, i.e. $S = \varphi(K^{alg})$, alors si $K \to L$ est une extension complète, on posera*

$$S(L^{alg}) := \varphi(L^{alg}). \tag{6}$$

d'après la propriété d'uniformité, cette définition ne dépendra pas de φ mais seulement de S.

Remarque 0.5.10. *Dans [Lip93, 3.8.2], la preuve n'est faite que pour l'élimination des variables de $(K^{alg})^\circ$ mais on peut vérifier qu'elle marche bien pour les variables de Γ.*

Définition 0.5.11. *Si $S \subset \left((K^{alg})^\circ\right)^m \times \Gamma^n$ et $T \subset \left((K^{alg})^\circ\right)^{m'} \times \Gamma^{n'}$ sont sous-analytiques, on dit que $f : S \to T$ est une application sous-analytique si son graphe, Graph$(f) \subset \left((K^{alg})^\circ\right)^{m+m'} \times \Gamma^{n+n'}$ est sous-analytique.*

Géométrie quasi-affinoïde

Expliquons comment géométriser cette théorie(cf. [LR00b, 2] et [LR00c]).

Théorème 0.5.12 (Nullstellensatz)**.** *[LR00d, 4.1.1] Le spectre maximal Max$(S_{m,n})$ est en bijection avec les orbites des $m+n$-uplets $(x_1,\dots,x_m,y_1,\dots,y_n) \in \left((K^{alg})^\circ\right)^m \times \left((K^{alg})^{\circ\circ}\right)^n$ modulo l'action de Gal(K).*

Définition 0.5.13. *Une **algèbre quasi-affinoïde** est une k-algèbre isomorphe à un quotient $A = S_{m,n}/I$. Dans ce cas, $X :=$ Max(A) est appelé un espace quasi-affinoïde.*

Une partie semi-analytique de X est une combinaison booléenne d'inégalités $\{|f| \le |g|\}$ avec $f,g \in A$.

Pour simplifier les choses, oublions l'action de Gal(k^{sep}/k), et assimilons une partie sous-analytique à une partie $S \subset \left((K^{\mathrm{alg}})^\circ\right)^{m+n}$, (c'est le cas si K est algébriquement clos).

Si $I = (f_1,\dots,f_N)$ est un idéal de $S_{m,n}$, alors on peut assimiler X aux $m+n$-uplets $(x,y) \in \left((K^{\mathrm{alg}})^\circ\right)^m \times \left((K^{\mathrm{alg}})^{\circ\circ}\right)^n$ tels que $f_i(x,y) = 0$ pour tout i. Si $A = S_{m,n}/I$ est une algèbre quasi-affinoïde et $f,g \in A$, on pose [LR00d, 5.3.1]

$$A\langle f/g\rangle := S_{m+1,n}/(I, f - X_{m+1}g) \quad \text{et}$$
$$A[[f/g]]_s := S_{m,n+1}/(I, f - (\rho_{n+1})g).$$

Ces définitions sont indépendantes de la présentation de A ([LR00d, 5.2.6].

Définition 0.5.14.

1. *On définit inductivement les **anneaux de fractions généralisés** A au dessus de $k\langle X_1,\dots,X_n\rangle$ comme une algèbre quasi-affinoïde A et un morphisme $f : k\langle X_1,\dots,X_n\rangle \to A$:*

– $k\langle X_1, \ldots, X_n\rangle \xrightarrow{id} k\langle X_1, \ldots, X_n\rangle$ *est un anneau de fractions généralisé.*
– *si* $k\langle X_1, \ldots, X_n\rangle \to A$ *est un anneau de fractions généralisé, et* $f, g \in A$*, alors* $k\langle X_1, \ldots, X_n\rangle \to A\langle f/g\rangle$ *est un anneau de fractions généralisé.*
De même, $A[[f/g]]_s$ *est un anneau de fractions généralisé.*

2. *Soit* $k\langle X_1, \ldots, X_n\rangle \to A$ *un anneau de fractions généralisé. On définit inductivement le domaine de A, noté Dom(A) comme une partie de Max(A) :*
– *Si* $A = k\langle X_1, \ldots, X_n\rangle$*, alors* $Dom(A) = Max(k\langle X_1, \ldots, X_n\rangle) = \left((K^{alg})^\circ\right)^n/Gal(K)$.
– *Si A est un anneau de fractions généralisé,* $f, g \in A$,

$$Dom(A\langle f/g\rangle) := \{x \in MaxA\langle f/g\rangle \mid |f(x)| \leq |g(x)| \neq 0\}$$

$$Dom(A[[f/g]]_s := \{x \in MaxA[[f/g]]_s \mid |f(x)| < |g(x)|\}$$

Remarque 0.5.15. *Si A est un anneau de fractions généralisé,* $\varphi : Max(A\langle f/g\rangle) \to Max(A)$ *induit une injection sur* $Dom(A\langle f/g\rangle)$ *(de même pour* $A[[f/g]]_s$*). Cependant,* φ *n'est pas bijective au-dessus des points de Max(A) où f et g s'annulent, les fibres étant alors des disques fermés (ouverts si l'on considère* $A[[f/g]]_s$*).*

Pour résumer, $Max(A) \to \left((K^{alg})^\circ\right)^n/Gal(K)$*, identifie Dom(A) avec son image dans* $\left((K^{alg})^\circ\right)^n/Gal(K)$.

Proposition 0.5.16 ([LR00b]). *Soit* $S \subset \left((K^{alg})^\circ\right)^n$*. On a équivalence entre*

1. *S est sous-analytique.*
2. *Il existe* $A_1, \ldots, A_m$ *des anneaux de fractions généralisés de* $k\langle X_1, \ldots, X_n\rangle$*, pour tout i une partie semi-analytique* S_i *de* $Max(A_i)$ *contenue dans* $Dom(A_i)$ *telle que* $S = \cup_{i=1}^m S_i$.
3. *Il existe* $A_1, \ldots, A_m$ *des anneaux de fractions généralisés de* $k\langle X_1, \ldots, X_n\rangle$*, pour tout i une partie Zariski-fermée* S_i *de* $Max(A_i)$ *telle que* $S = \cup_{i=1}^m (Dom(A_i) \cap S_i)$.

Signalons que l'on peut généraliser tout ce qui a été fait aux parties d'un espace quasi-affinoïde $X = \mathrm{Max}(A)$ où A est une algèbre quasi-affinoïde.

Définition 0.5.17. *Si* $X = Max(A)$ *est un espace quasi-affinoïde,* $A \simeq S_{m,n}/I$ *une présentation de A, de sorte que l'on assimile X à une partie de* $\left((K^{alg})^\circ\right)^m \times \left((K^{alg})^\circ\right)^n$*. Soit une partie* $S \subset X$*. On dira que S est sous-analytique si c'est une partie sous-analytique de* $\left((K^{alg})^\circ\right)^{m+n}$.

On peut s'assurer que cette définition ne dépend pas de la présentation de A.

0.5.2 Une extension du foncteur Sp

Ce qui suit découle directement des résultats de [LR00d], mais n'est pas vraiment énoncé ainsi.

Considérons une algèbre k-quasi-affinoïde $A = S_{m,n}/I$. On va expliquer comment lui associer un espace k-rigide, que l'on notera $X = \mathrm{Sp}(A)$. On fixe un $\pi \in k^{\circ\circ}$, $\pi \neq 0$. On pose alors pour $i \in \mathbb{N}^*$,

$$X_j = \mathrm{Sp}(k\{X_1, \ldots, X_m, (|\pi|^{\frac{1}{j}})^{-1}\rho_1, \ldots, (|\pi|^{\frac{1}{j}})^{-1}\rho_n\}/I).$$

Ce sont des espaces affinoïdes vérifiant $X_1 \subset X_2 \subset \ldots$ les inclusions correspondant à des inclusions de domaines de Weierstrass. On définit alors l'espace X ainsi :

$$X = \mathrm{Sp}(A) := \bigcup_{j\geq 1} X_j.$$

Ensemblistement, le Nullstellensatz pour les algèbres quasi-affinoïdes [LR00d, 4.1.1] nous assure que $X = \mathrm{Max}(A)$, l'ensemble des idéaux maximaux de A. De plus, on a pour tout entier $j \geq 1$ un morphisme $A \to \Gamma(X_j, \mathcal{O}_{X_j})$ qui induit un morphisme $A \to \Gamma(X, \mathcal{O}_X)$.

Il nous faut comprendre pourquoi on obtient bien quelque chose de fonctoriel (et qui ne dépend pas de la présentation choisie). On peut montrer que si $B = S_{p,q}/J$ est une autre algèbre quasi-affinoïde, et $Y = \mathrm{Sp}(B)$ obtenu comme la réunion des Y_j, et si $\varphi : A \to B$ est un morphisme d'algèbres quasi-affinoïdes, alors il existe un unique morphisme d'espaces rigides $Y \to X$ rendant commutatif

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ \Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(Y, \mathcal{O}_Y) \end{array} \tag{7}$$

Pour ce faire, si l'on note

$$B_j = k\{X_1, \ldots, X_p, (|\pi|^{\frac{1}{j}})^{-1}\rho_1, \ldots, (|\pi|^{\frac{1}{j}})^{-1}\rho_q\}/J$$

l'algèbre affinoïde associée à Y_j, on considère la composée.

$$\varphi_j : A \to B \to B_j.$$

Toujours grâce au Nullstellensatz, cela induit une application $Y_j = \mathrm{Max}(B_j) \to \mathrm{Max}(A) = X$. Si $y \in Y_j$ et x est son image dans X par cette application, alors pour tout $i = 1 \ldots n$, $|\varphi_j(\rho_i)(y)| = |\rho_i(x)| < 1$ d'après [LR00d, 5.1.8] appliqué à ρ_i. Alors, d'après le principe du maximum appliqué à l'algèbre affinoïde B_j, il s'ensuit que $\|\varphi_j(\rho_i)\| < 1$. Il existe donc un indice $l \in \mathbb{N}^*$ qui dépend [10] de j tel que $\|\varphi_j(\rho_i)(y)\| \leq ||pi|^{\frac{1}{l}}$. Cela implique que l'image de $Y_j = \mathrm{Max}(B_j) \to \mathrm{Max}(A) = X$ tombe dans X_l. Alors d'après la propriété universelle des domaines quasi-affinoïdes [LR00d, 5.3.5] (X_l est un domaine quasi-affinoïde de $\mathrm{Max}(A)$) on peut factoriser de manière unique

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_l & \underset{\exists!}{\longrightarrow} & B_j \end{array}$$

Cela induit un morphisme d'espaces affinoïdes $Y_j \to X_l$. Ces morphismes sont donc bien compatibles quand on fait varier j, et permettent de définir un unique morphisme d'espaces rigides $Y \to X$ vérifiant (7).

On a ainsi défini un foncteur

$$\left\{ \begin{array}{c} \text{Algèbres} \\ \text{quasi-affinoïdes} \end{array} \right\} \longrightarrow \text{Espaces rigides} \tag{8}$$

$$A \simeq S_{m,n}/I \longmapsto \mathrm{Sp}(A) = \bigcup_{j \geq 1} \mathrm{Sp}(A_j)$$

10. En utilisant le théorème d'élimination des quantificateurs [Lip93, 3.8.1], on peut montrer qu'il existe même une constante $\kappa \in \mathbb{N}^*$ telle que $l = \kappa j$ marche, i.e. que Y_j tombe dans $X_{\kappa j}$ pour tout j.

qui prolonge le foncteur Sp défini pour les algèbres k-affinoïdes. Ce foncteur

$$\left\{ \begin{array}{c} \text{Algèbres} \\ \text{k-affinoïdes} \end{array} \right\} \longrightarrow \text{Espaces rigides}$$

$$A \longmapsto \mathrm{Sp}(A)$$

est pleinement fidèle [BGR84, 9.3.1]. En revanche son extension (8) aux algèbres quasi-affinoïdes n'est pas pleinement fidèle en général. Il est fidèle mais pas plein. Considérons par exemple $X = \mathrm{Sp}(S_{0,1})$ qui n'est autre que la boule unité ouverte. Par définition, $S_{0,1}^{\circ} \subset k^{\circ}[[\rho_1]]$ et cette inclusion est stricte en général. Par exemple, si l'on prend une suite $\{a_i\}_{i\in\mathbb{N}}$ telle que

$$\begin{aligned} a_i &\in k^{\circ\circ} \text{ et} \\ |a_i| &\xrightarrow[i\to\infty]{} 1 \end{aligned} \tag{9}$$

alors $f = \sum_{i\in\mathbb{N}} a_i \rho_1^i \in k^{\circ}[[\rho_1]] \setminus S_{0,1}^{\circ}$, par exemple car f a une infinité de racines. La série f induit un morphisme $\varphi : X \to X$ qui ne s'obtient donc pas comme un morphisme $S_{0,1} \to S_{0,1}$. Remarquons que pour que les conditions (9) soient satisfaites, la valuation de k ne doit pas être discrète. Précisément, nous allons expliquer dans la prochaine section (proposition 0.5.18) que si k est de valuation discrète, le foncteur Sp restreint aux algèbres quasi-affinoïdes réduites est pleinement fidèle.

0.5.3 Le cas de la valuation discrète

On suppose k de valuation discrète, et on fixe π une uniformisante de k. Dans ce cas, comme on l'a expliqué,

$$S_{m,n} = k \otimes_{k^{\circ}} k^{\circ}\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_m]]$$

et les espaces rigides Sp(A) pour A une algèbre quasi-affinoïde que nous avons introduits dans la sous-section précédente apparaissent dans la littérature sous des noms et dans des contextes différents [Ber96b, Ber96a, Con99, dJ95, RZ96, Kap10].

Soit $A \simeq S_{m,n}/(f_1, \ldots, f_p)$ une algèbre quasi-affinoïde. Quitte à normaliser les f_i, on peut supposer que $f_i \in k^{\circ}\{X_1, \ldots, X_m\}[[\rho_1, \ldots, \rho_m]]$ et alors

$$A \simeq k \otimes_{k^{\circ}} \Big(k^{\circ}\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_m]]/(f_1, \ldots f_p)\Big).$$

Le foncteur de Berthelot

On munit $k^{\circ}\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_m]]$ de la topologie $(\pi, \rho_1, , \ldots, \rho_n)$-adique, qui est alors un anneau noethérien adique (i.e séparé et complet). Suivant [Ber96a, 1], on appelle **k°-algèbre spéciale** une algèbre adique de la forme $k^{\circ}\langle X_1, \ldots, X_m\rangle[[\rho_1, \ldots, \rho_m]]/I$ dont un idéal de définition est $(\pi, \rho_1, , \ldots, \rho_n)$. On note $\mathfrak{X} = \mathrm{Spf}(A)$ le k°-schéma formel adique associé. Si A est une k°-algèbre adique et $\mathbf{a}$ un idéal de définition, on a les équivalences suivantes [Ber96a, 1] :

– A est une k°-algèbre spéciale.
– Pour tout $n \in \mathbb{N}$, $A/\mathbf{a}^n$ est de type fini sur $\tilde{k}$.
– $A/\mathbf{a}^2$ est de type fini sur $\tilde{k}$.
– A est noethérien et $A/\mathbf{a}$ est de type fini sur $\tilde{k}$.

Chez [Ber96b, dJ95, Con99], on considère $\mathfrak{X}$ un k°-schéma formel affine adique noethérien tel que $\mathfrak{X}_s$ est de type fini sur $\tilde{k}$. Cela équivaut donc à dire que $\mathfrak{X} = \mathrm{Spf}(A)$ pour A une k°-algèbre spéciale. Berthelot [Ber96b, 0.2.6] a le premier remarqué que l'on pouvait associer fonctoriellement à un tel $\mathfrak{X}$ un espace rigide noté[11] $\mathfrak{X}^{\mathrm{rig}}$. Plus généralement, en notant FS_{k° la catégorie des k°-schémas formels adiques $\mathfrak{X}$ localement noethériens et tels que $\mathfrak{X}_s$ est localement de type fini sur $\tilde{k}$, on obtient un foncteur $FS_{k^\circ} \overset{\mathrm{rig}}{\to} \mathrm{Rig}_k$. Nous renvoyons à [Ber96b, 0.2.6] et [dJ95, 7.1] pour les détails de cette construction. Ce qui nous importe est que le diagramme suivant est commutatif :

$$\begin{array}{ccc} \left\{ \begin{array}{c} k^\circ\text{-algèbres} \\ \text{spéciales} \end{array} \right\} & \xrightarrow{\mathrm{Spf}} & \mathrm{FS}_{k^\circ} \\ \downarrow {\scriptstyle k\otimes_{k^\circ}} & & \downarrow {\scriptstyle \mathrm{rig}} \\ \left\{ \begin{array}{c} \text{algèbres} \\ \text{quasi-affinoïdes} \\ \text{sur } k \end{array} \right\} & \xrightarrow[\mathrm{Sp}]{} & \mathrm{Rig}_k \end{array} \tag{10}$$

La commutativité de ce diagramme découle directement (voir par exemple le début de [dJ95, 7.1]) des définitions de rig et Sp. Cela nous donne au passage une définition du foncteur rig.

Le foncteur $k\otimes_{k^\circ}$ n'est pas fidèle, mais il est surjectif sur les objets. Le foncteur Spf est lui pleinement fidèle. Le foncteur rig n'est pas pleinement fidèle. Nous voulons insister sur le fait que

Proposition 0.5.18 ([Kap]). *La restriction de Sp aux algèbres quasi-affinoïdes réduites*

$$\left\{ \begin{array}{c} \textit{algèbres} \\ \textit{quasi-affinoïdes} \\ \textit{sur } k \textit{ réduites} \end{array} \right\} \xrightarrow{Sp} Rig_k$$

est pleinement fidèle.

Démonstration. Ce résultat découle de la proposition 0.5.20 ci dessous. □

Nous allons montrer la proposition 0.5.20 en nous appuyant principalement sur le résultat suivant :

Théorème 0.5.19 (de Jong, [dJ95, 7.4.1]). *Soit A une k°-algèbre spéciale plate et normale, et soit $\mathfrak{X} = Spf(A)$. L'application naturelle $A \to \Gamma(\mathfrak{X}^{rig}, \mathcal{O}^\circ_{\mathfrak{X}^{rig}})$ est un isomorphisme où on a noté $\mathcal{O}^\circ_{\mathfrak{X}^{rig}}$ le faisceau des fonctions analytiques dont la valeur absolue est bornée par* 1.

En tensorisant par k, on en tire le corollaire suivant, dont la preuve nous a été donnée par C. Kappen :

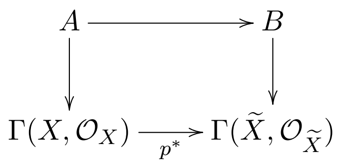
Proposition 0.5.20 ([Kap]). *Soit $\mathcal{A}$ une algèbre k-quasi-affinoïde réduite et notons $X = Sp(\mathcal{A})$ l'espace rigide associé. Le morphisme naturel*

$$\mathcal{A} \to \{f \in \Gamma(X, \mathcal{O}_X) \mid \|f\|_{sup} \leq 1\}$$

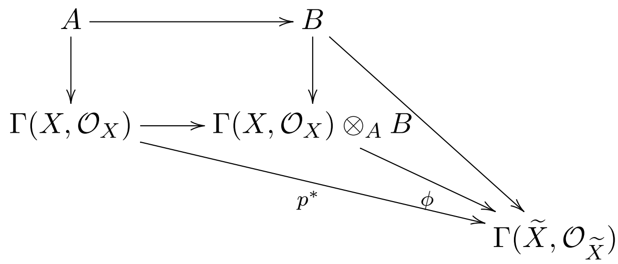
est un isomorphisme.

11. Dans [Ber96b] il est noté $\mathfrak{X}_k$. Nous optons ici pour la notation de [Con99, dJ95].

Démonstration. On considère A une k°-algèbre spéciale réduite et plate telle que $\mathcal{A} = k \otimes_{k^\circ} A$, de sorte que A s'identifie à un sous-anneau de $\mathcal{A}$ (pour obtenir un tel A, on commence par choisir une k°-algèbre spéciale A' telle que $\mathcal{A} = A' \otimes_{k^\circ} k$, puis on définit A comme le quotient de A' par son l'idéal de torsion). Remarquons que si A est normale, le résultat découle du théorème de de Jong. On va expliquer comment s'y ramener.
On note B la normalisation de A. D'après les résultats d'excellence de [Val75, Val76], B est une k°-algèbre spéciale. Alors [Con99, 2.1.3] $\widetilde{X} = \mathrm{Sp}(k \otimes_{k^\circ} B) = (\mathrm{Spf}(B))^{\mathrm{rig}}$ est la normalisation de X au sens de [Con99, 2.1], et on note $p : \widetilde{X} \to X$ le morphisme de normalisation. On a alors le diagramme



qui s'insère dans la diagramme suivant :



Soit donc $f \in \Gamma(X, \mathcal{O}_X)$ une fonction bornée. Alors $p^*(f)$ est bornée sur $\widetilde{X}$ aussi, et comme B est normal, d'après le théorème de de Jong, $p^*(f)$ provient d'un élément de B.

On remarque d'abord que ϕ est injective, ce qui découle des propriétés de normalisations. Ainsi comme par hypothèse $\phi((f) \otimes 1) = p^*(f) = \phi(1 \otimes b)$ on en déduit que $f \otimes 1 = 1 \otimes b$.

On remarque ensuite que $\Gamma(X, \mathcal{O}_X)$ est fidèlement plat sur A. Pour ce faire, notons $A_j = k\{X_1, \ldots, X_m, (|\pi|^{\frac{1}{j}})^{-1}\rho_1, \ldots, (|\pi|^{\frac{1}{j}})^{-1}\rho_n\}/I$ et considérons le foncteur

$$\begin{array}{rccc} \theta : & A\text{-modules finis} & \to & \Gamma(X, \mathcal{O}_X)\text{-modules} \\ & M & \mapsto & \varprojlim_{j \geq 1} A_j \otimes_A M \end{array}$$

Ce foncteur est exact. Pour le montrer, on utilise le fait que les A_j sont plats sur A et conclut comme dans la preuve [Bos77, 2.1]. On en déduit que si M est un A-module fini, $\theta(M) = \Gamma(X, \mathcal{O}_X) \otimes_A M$ (car c'est vrai pour $M = A$ et θ est exact). Comme A est noethérien, $\Gamma(X, \mathcal{O}_X)$ est donc une A-algèbre plate. Il est alors facile de voir qu'elle est fidèlement plate (en utilisant le Nullstellensatz sur A).

Puis on considère B comme un A-module, et $A \subset B$ comme un sous A-module de B (vrai car A est supposée réduite ; c'est le seul moment où on utilise cette hypothèse). Comme $\Gamma(X, \mathcal{O}_X)$ est fidèlement plat sur A, $\Gamma(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) \otimes_A A$ s'identifie à un sous-module de $\Gamma(X, \mathcal{O}_X) \otimes_A B$ et via cette identification :

$$(\Gamma(X, \mathcal{O}_X) \otimes_A A) \cap \Big((\Gamma(X, \mathcal{O}_X) \otimes_A B) \cap (A \otimes_A B) \Big) = A \otimes_A A = A.$$

Cela découle du résultat [Bou98, I 3.5 Prop 10(ii)] en vertu duquel si $C \to C'$ est un morphisme d'anneaux fidèlement plat, N un C-module et $N' \subset N$ un sous C-module, alors $N' \otimes_C C' \cap N = N'$.

En particulier on en déduit les égalités $f \otimes 1 = 1 \otimes b \in A \otimes_A A$. Donc $f \in A$. □

Si A n'est pas réduite, le résultat n'est plus vrai. Prenons par exemple, $A = S_{1,1}/(\rho_1^2)$ qui n'est pas réduite. On vérifie que la fonction $f = \sum_{j\geq 0} \pi^{-j}\rho_1^{j^2} X_1$ est une fonction bornée (car nulle en tout point) de $\Gamma(X, O_X)$ qui ne provient pas de A.

0.6 Cohomologie étale en géométrie non archimédienne

0.6.1 Cohomologie étale des espaces k-analytiques

Berkovich définit [Ber93, 3.3.4] la notion de morphisme étale d'espaces k-analytiques comme un morphisme quasi-fini non-ramifié et plat ; ils sont stables par composition, changement de base... Une immersion ouverte est étale, mais l'immersion d'un domaine affinoïde n'est pas en général étale (car pas quasi-fini). La site étale sur X est alors défini comme dans le cas des schémas. C'est la site sur la catégorie $\text{Ét}(X)$ des morphismes étales au dessus de X engendré par les recouvrements $\{U_i \xrightarrow{f_i} U\}_i \in I$ tels que $\cup_{i\in I} f_i(U_i) = U$. On définit alors la catégorie des faisceaux étales sur X, $S(X)$ et on définit pour $F \in S(X)$ des groupes de cohomologie étale que l'on notera $H^q(X, F)$. Si $s \in \Gamma(X, F)$ est une section de F, on peut définir le support [Ber93, p.87] de s, noté $\text{supp}(s) \subset X$ qui est un fermé de X. Alors, si X est Hausdorff, le foncteur

$$\begin{array}{rccc} \Gamma_c : & S(X) & \to & \text{Ab} \\ & F & \mapsto & \{s \in \Gamma(X, F) \mid \text{supp}(s) \text{ est compact}\} \end{array}$$

est un foncteur exact à gauche dont on notera $H_c^q(X, F)$ le q-ième foncteur dérivé. Quand l'espace X est compact, on obtient donc un isomorphisme canonique :

$$H_c^q(X, F) \xrightarrow{\sim} H^q(X, F).$$

Parmi les nombreuses propriétés démontrées dans [Ber93], citons

– *Théorèmes de comparaison.*Si $\mathcal{X}$ est un k-schéma de type fini séparé et F un faisceau étale de torsion sur $\mathcal{X}$ alors [Ber93, 7.1] $H_c^q(\mathcal{X}, F) \xrightarrow{\sim} H_c^q(\mathcal{X}^{\text{an}}, F^{\text{an}})$, si de plus F est constructible et de torsion première à la caractéristique de k alors d'après [Ber95], $H^q(\mathcal{X}, F) \xrightarrow{\sim} H^q(\mathcal{X}^{\text{an}}, F^{\text{an}})$.
– Une dualité de Poincaré pour les morphismes lisses, des théorèmes de changement de base propre et lisse, une formule de Künneth.

0.6.2 Cohomologie étale des espaces adiques

R. Huber a introduit et étudié une autre catégorie d'espaces, les espaces adiques. Leur construction peut se résumer ainsi.

Étape 1. Dans [Hub93] est définie la notion d'anneaux affinoïdes [Hub93, 3], il s'agit d'une paire $A = (A^\triangleright, A^+)$ où $A^\triangleright$ est un anneau topologique d'un certain type (appelé anneau f-adique [Hub93, 1]) et A^+ un sous-anneau de $A^\triangleright$ ouvert et intégralement clos. On associe à un anneau affinoïde A un espace topologique $\text{Spa}(A)$, l'ensemble des valuations continues de $A^\triangleright$ majorées par 1 sur A^+ dont on montre que c'est un espace spectral (de manière équivalente, isomorphe à $\text{Spec}(B)$ pour un anneau B [Hoc69]).

Étape 2. Dans [Hub94], les espaces $X = \text{Spa}(A)$ sont munis d'un préfaisceau [12] $\mathcal{O}_X$, qui est un faisceau [Hub94, 2.2] quand $A^\triangleright$ est un anneau fortement noethérien, i.e. quand $A^\triangleright\{T_1, \ldots, T_n\}$ est noethérien pour tout n.

12. qui n'est en général pas un faisceau, voir [Hub94, p.520-521] pour un contre-exemple.

Étape 3. On définit $\mathcal{V}$ comme la catégorie dont les objets sont des triplets $(X, \mathcal{O}_X, (v_x)_{x\in X})$ où $(X, \mathcal{O}_X)$ est un espace localement annelé en anneaux topologiques complets et pour tout x, v_x est une valuation sur $O_{X,x}$. Quand $\mathcal{O}_{\mathrm{Spa}(A)}$ est un faisceau, $\mathrm{Spa}(A)$ devient naturellement un objet de $\mathcal{V}$, appelé espace adique affinoïde.

Étape 4. Un espace adique est alors un objet de $\mathcal{V}$ localement isomorphe à un espace adique affinoïde.

Ils forment une classe très générale d'espaces. Si k est un corps non archimédien, il existe un foncteur $r_k : \mathrm{Rig}_k \to \{\text{Espaces adiques}\}$ qui est pleinement fidèle [Hub94, 4.3] et un foncteur

$$t : \quad \begin{array}{c}\text{Schémas formels}\\ \text{local}^t\text{ noethériens}\end{array} \longrightarrow \text{Espaces adiques}$$

qui est également pleinement fidèle [Hub94, 4.1]. Signalons que si $\mathcal{A}$ est une algèbre k-affinoïde, l'espace adique associé à $\mathrm{Sp}(\mathcal{A})$ par r_k est $\mathrm{Spa}(\mathcal{A}, \mathcal{A}^\circ)$, la paire $(\mathcal{A}, \mathcal{A}^\circ)$ étant bien un anneau affinoïde, et $\mathcal{A}$ étant fortement noethérien.

Dans [Hub96, 1.6.5] est définie la notion de morphisme étale entre espace adiques [13], ce qui permet de définir [Hub96, 2.1,2.3] le site étale d'un espace adique X comme pour les schémas, et pour F un faisceau étale sur X on a de nouveau des groupes de cohomologie $H^q(X, F)$. Pour définir une cohomologie à support propre, il faut en revanche procéder comme pour les schémas en compactifiant les morphismes. Il est montré que tout morphisme d'espaces adique *raisonnable* $f : X \to Y$ a une compactification [Hub96, 5.1.11] ce qui permet de définir le foncteur $R^+ f_!$ [Hub96, 5.4.3] pour ces morphismes *raisonnables* et même un foncteur $Rf_!$ pour des morphismes *très raisonnables* [Hub96, 5.5.4]. Si $f : X \to Y$ est un morphisme d'espaces rigides sur k, alors si f est séparé et taut [14] (cf [Hub96, définition 5.6.6]) le morphisme d'espaces adiques associé est *raisonnable.*

Une dualité de Poincaré, des théorèmes de changement de base propre et lisse sont démontrés. Si X est un espace k-analytique Hausdorff, F un faisceau étale sur X, on peut lui associer un espace rigide X^{rig} puis un espace adique X^{ad} et un faisceau étale X^{ad}. On dispose du théorème de comparaison [Hub96, 8.3]

$$H^q(X, F) \simeq H^q(X^{\mathrm{ad}}, F^{\mathrm{ad}}).$$

En revanche, les groupes de cohomologie à support compact $H^q_c(X, F)$ ne sont pas isomorphes aux groupes de cohomologie à support propre $H^q_c(X^{\mathrm{ad}}, F^{\mathrm{ad}})$ (par exemple quand X est le disque unité, cf remarque 2.5.1).

0.6.3 Théorèmes de constructibilité déjà existants

Dans cette partie, on va supposer que tous nos faisceaux F sont de torsion première à $\mathrm{car}(\tilde{k})$ et sauf mention contraire, on supposera k algébriquement clos.

On veut présenter, dans l'ordre chronologique de leur apparition, les résultats de constructibilité qui existent déjà.

1. Si $\varphi : \mathcal{Y} \to \mathcal{X}$ est un morphisme séparé de $\mathcal{A}$ schémas de type finis, et F un faisceau de torsion sur $\mathcal{Y}$, alors [Ber93, 7.1.4]

$$\left(R^q\varphi_!(F)\right)^{\mathrm{an}} \xrightarrow{\sim} R^q(\varphi^{\mathrm{an}})_! F^{\mathrm{an}}.$$

13. Les morphismes étales entre espaces adiques diffèrent de ceux définis par Berkovich. Par exemple, dans le monde des espaces adiques l'immersion d'un domaine affinoïde est étale (car c'est une immersion ouverte dans la catégorie des espaces adiques), alors que ce n'est pas le cas chez Berkovich.

14. En pratique, la plupart des morphismes de la vie courante sont taut.

Si $\varphi : \mathcal{Y} \to \mathcal{X}$ est un morphisme de k schémas de type finis, et F un faisceau constructible sur $\mathcal{Y}$ de torsion première à car(k), alors [Ber95]

$$\left(R^q\varphi_*(F)\right)^{\mathrm{an}} \xrightarrow{\sim} R^q(\varphi^{\mathrm{an}})_*F^{\mathrm{an}}.$$

Pour ces énoncés, on n'a pas besoin de k algébriquement clos.

2. Si X est un domaine affinoïde de $\mathcal{X}^{\mathrm{an}}$ où $\mathcal{X}$ est un k-schéma de type fini, et F un faisceau constructible sur $\mathcal{X}$, alors [Ber94, 5.5] les groupes $H^q(X, F^{\mathrm{an}}) \simeq H^q_c(X, F^{\mathrm{an}})$ sont finis, et donc les groupes $H^q(X^{\mathrm{ad}}, F^{\mathrm{ad}})$ aussi. Ce résultat découle d'un théorème de comparaison de cycles évanescents [Ber94, 5.3]

$$R^q\psi_\eta(\mathcal{F}) \xrightarrow{\sim} R^q\psi(\widehat{\mathcal{F}}). \tag{11}$$

3. Le théorème [Ber96a, 3.1] généralise la formule (11) à une classe plus importante de schémas formels, obtenus en complétant un k°-schéma de type fini le long d'un sous-schéma $\mathcal{Y}$ de $\mathcal{X}_s$.
4. Dans le même genre, Huber définit une notion de **faisceau constructible** pour les espaces adiques [Hub96, 2.7] et montre que si X est un domaine affinoïde de $\mathcal{X}^{\mathrm{ad}}$, et si F est un faisceau constructible de $\mathbb{Z}/\ell^n\mathbb{Z}$-modules sur X, les groupes $H^q(X, F)$ et $H^q_c(X, F)$ sont finis [Hub96, 6.1.1,6.2.1].

   De plus si $f : X \to Y$ est un morphisme *lisse et quasi-compact* d'espaces adiques analytiques, et F un faisceau constructible sur X, alors les faisceaux $R^qf_!F$ sont constructibles sur Y, voir [Hub96, 6.2.2] plus la remarque dans [Hub07, 7.1].
5. Si X est un espace adique analytique, la catégorie des **faisceaux quasi-constructibles** est introduite dans [Hub98b, 1.1] et il est montré [Hub98b, 2.1] que si $f : X \to Y$ est un morphisme d'espaces adiques provenant d'un morphisme d'espaces rigides, séparé et quasi-compact, si k est de caractéristique 0, et $\dim(Y) \leq 1$ alors si F est quasi-constructible sur X, les faisceaux $R^qf_!F$ sont quasi-constructibles sur Y.

   Cela implique que si X est l'espace adique associé à un espace k-affinoïde et si car(k) = 0, alors les groupes $H^q_c(X, F)$ sont finis. Cela généralise le résultat 4.
6. Si X est un espace adique, la classe des **faisceaux oc-quasi-constructibles** sur X est définie dans [Hub98a, 1.2] et il est montré [Hub98a, 2.2] que si $f : X \to Y$ est un morphisme d'espaces adiques provenant d'un morphisme d'espaces rigides, quasi-séparé et quasi-compact, si k est de caractéristique 0, et $\dim(Y) \leq 1$ alors si F est oc-quasi-constructible sur X, les faisceaux R^qf_*F sont oc-quasi-constructibles sur Y.

   Cela implique que si X est l'espace adique associé à un espace k-affinoïde et si car(k) = 0, alors les groupes $H^q(X, F)$ sont finis. Cela généralise les résultats 4 et 2.
7. Pour X un espace adique, une classe de faisceaux $\mathcal{C}(X)$ qui contient la classe des faisceaux quasi-constructibles est construite dans [Hub07, 4.2], qui vérifie une certaine stabilité [Hub07, 5.1] par $R^qf_!$, toujours pour f quasi-compact et $\dim(Y) \leq 1$, mais sans restriction sur la caractéristique de k.
8. J. Poineau a montré [Poi08] que si X est un espace k-affinoïde irréductible et $f \in \mathcal{A}$, alors les domaines de Laurent définis pas $|f| \geq \varepsilon$ sont irréductibles pour ε assez petit.
9. Dans [Ber13] les résultats de [Ber94, Ber96a] sont généralisés. Il est prouvé [Ber13, 1.1.1] que si X est un espace k-affinoïde et F un faisceau provenant d'un faisceau constructible sur Spec($\mathcal{A}$), alors les groupes $H^q(X, F)$ (et donc aussi $H^q_c(X, F)$) sont finis. Ce résultat utilise un théorème d'uniformisation récent de Gabber [ILO12].

De plus, un théorème [Ber13, 3.1.1] de constructibilité pour les faisceaux de cycles évanescents $R^q\psi_\eta F$ est prouvé, qui est valable quand k est de valuation discrète et $\mathfrak{X} = \mathrm{Spf}(A)$ avec A un k°-algèbre spéciale.

Chapter 1

Overconvergent subanalytic sets

Introduction

Let us start with a definition :

Definition 1.0.1. Let $\mathcal{P}$ be the data, for each k-affinoid space X, of a family $\mathcal{P}_X$ of subsets of X. If S is a subset of a k-affinoid space X, we will say that S satisfies the property $\mathcal{P}$ if $S \in \mathcal{P}_X$. We will say that

- The property $\mathcal{P}$ is a **G-local property** if for all k-affinoid spaces X and any subset S of X, S satisfies the property $\mathcal{P}$ if and only if for all finite affinoid covers $\{X_i\}$ of X, $S \cap X_i$ satisfies the property $\mathcal{P}$ relatively to X_i (i.e. $S \cap X_i \in \mathcal{P}_{X_i}$).
- the property $\mathcal{P}$ is a **local property** if for all affinoid spaces X and any subset S of X, $S \in \mathcal{P}_X$ if and only if for all $x \in X$, there exists an affinoid neighbourhood U of x such that $S \cap U \in \mathcal{P}_U$.

Note that using the compactness of affinoid spaces, saying that $\mathcal{P}$ is a local property is equivalent to requiring that for all k-affinoid spaces X and any $S \subseteq X$, S satisfies $\mathcal{P}$ if and only if for any finite affinoid covering $\{X_i\}$ of X such that $\{\overset{\circ}{X_i}\}$ is also a covering of X, then $S \cap X_i \in \mathcal{P}_{X_i}$.

As a consequence, if $\mathcal{P}$ is also a G-local property, then it is a local property.

Example 1.0.2. A consequence of Kiehl's theorem [BGR84, 9.4.3] is that the class of Zariski-closed subsets of affinoid spaces defines a class which is G-local.

Organisation of the chapter

In section 1.1, we define *constructible data* of X, in order to define overconvergent constructible subsets. Note that we do not assume that k is algebraically closed contrary to [Sch94a]. In section 1.1.2 we introduce overconvergent subanalytic subsets. In section 1.1.3 we have tried to carefully treat Weierstrass division, trying to be as general as possible (namely our results hold for an arbitrary ultrametric Banach algebra, and an arbitrary radius of convergence). In section 1.1.4 we prove that overconvergent constructible and overconvergent subanalytic subsets are the same. The proof of this result which appears in [Sch94a], is here simplified by the use of Berkovich spaces: in particular, the quite technical section 2 of [Sch94a] *A combinatorial lemma* is replaced by a simple compactness argument (see theorem 1.1.38). In 1.1.5 we try to handle the following problem: how to pass from a definition that works only for k-affinoid spaces to a more local definition, with the hope that in the affinoid case the local and the global definitions would coincide. As we said earlier, trying to do this with the G-topology will not work. If in contrary we do this with the Berkovich topology, the definitions will be compatible. This can be summed up by:

Theorem. *Let S be a subset of the strictly k-affinoid space X. The following properties are equivalent:*

1. *S is overconvergent constructible.*
2. *S is overconvergent subanalytic.*
3. *There exists $r > 1$, an integer n, T a locally semianalytic subset of $X \times \mathbb{B}^n_r$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$ where $\pi : X \times \mathbb{B}^n \to X$ is the natural projection.*
4. *S is, locally for the Berkovich topology, an overconvergent constructible subset.*

(1) $\Leftrightarrow$ (2) is theorem 1.1.38, (1) $\Leftrightarrow$ (3) is corollary 1.1.48, and (1) $\Leftrightarrow$ (4) is proposition 1.1.44. In section 1.1.6, we explain how these results can be extended to k-affinoid spaces (by opposition to strictly k-affinoid spaces). In addition, in that case, we can allow the field k to be trivially valued.

In section 1.2, we give some counter-examples to erroneous statements of [Sch94a]. Precisely, in [Sch94a] five classes of subsets were defined: globally strongly subanalytic, globally strongly D-semianalytic, strongly subanalytic, locally strongly subanalytic and strongly D-semianalytic subsets. The three last classes were defined from the first two ones by adding "G-local" at some point. In [Sch94a] it was claimed that these five classes agree. We explain that this is not the case, namely from these five classes, the first two ones indeed agree, but not the last three ones, which are larger (see figure 1.1, p. 82). What the above theorem says is that if one replaces "G-local" by "locally for the Berkovich topology", the results of [Sch94a], for instance the theorem on p. 270, become true. Let us give one of the counter-examples that we study:

Example 1.0.3. Let $X = \mathbb{B}^2$ be the the closed bidisc, $0 < r < 1$ with $r \in \sqrt{|k^\times|}$, $f \in k\{r^{-1}x\}$ an analytic function whose radius of convergence is exactly r and such that $\|f\| < 1$. We then define

$$S = \{(x,y) \in \mathbb{B}^2 \mid |x| < r \text{ and } y = f(x)\}.$$

Then (see proposition 1.2.4) S is rigid semianalytic, but not overconvergent subanalytic. The Berkovich approach is here helpful since to prove this, we use a point η of the Berkovich bidisc which is not a rigid point, and some properties of its local ring $\mathcal{O}_{X,\eta}$.

Finally, in section 1.3 we correct the proof of [Sch94b] (which rested on the erroneous results of [Sch94a], and [Sch94c]) and restrict the hypothesis of it. Namely, we prove that when k is algebraically closed, and X is a good quasi-smooth (equivalently rig-smooth) strictly k-analytic space of dimension 2, then overconvergent subanalytic subsets are in fact locally semianalytic. Not only do we give a correct proof of this theorem, but moreover, this result is more general than the result of [Sch94b], where X was the bidisc and where is was assumed that the characteristic of k was 0.

Contribution of this chapter

We want to stress the fact that section 1 is highly inspired by the work of H. Schoutens. In particular, the definition we give of a constructible datum, and the resulting definition of an overconvergent constructible subset, is a *geometric* formulation of what is done in [Sch94a] concerning D-strongly semianalytic subsets. In particular, the proof of theorem 1.1.38 is very close to the proof of [Sch94a, Th 5.2]. We have however decided to include a proof of theorem 1.1.38 for three reasons. First, the compactness argument that we use in theorem 1.1.38 seems to us enlightening, and a way to see that Berkovich spaces are relevant in this context [1]. Secondly, we have the feeling that replacing the strongly D-semianalytic subsets of [Sch94a] by our overconvergent constructible subsets is more geometric and gives a better understanding of the situation. Finally, the mistakes in [Sch94a], that we explain in section 2, have the following consequences: most of the statements of [Sch94a] become false. For instance, [Sch94a, Theorem 5.2] is false as we prove in section 2. In this context it seemed to us relevant to write section 1.

The same remarks hold for section 3. A statement analogous to theorem 1.3.12 was claimed in [Sch94b]. However, in this article, it was assumed, and used in the proofs, that the five classes of subsets introduced in [Sch94a] were the same; but since we prove that this is not the case, the proofs of [Sch94b] are erroneous.

Finally, let us mention that another proof of theorem 1.1.38 has also been given in [CL11, 4.4.10].

1.1 Overconvergent subanalytic sets

With a few exceptions that will be specified, $\mathcal{A}$ will be a strictly k-affinoid algebra, and X the strictly k-affinoid space $\mathcal{M}(\mathcal{A})$.

1.1.1 Constructible data

Definition 1.1.1. Let X be a k-affinoid space whose k-affinoid algebra is $\mathcal{A}$. A subset S of X is called **semianalytic** if it is a finite boolean combination of sets of the form $\{x \in X \mid |f(x)| \leq |g(x)|\}$ where f and $g \in \mathcal{A}$ (by finite boolean combination, we mean finitely many use of the set-theoretical operators $\cap$, $\cup$ and ^{c}).

A subset of the form $\{x \in X \mid |f_i(x)| \bowtie_i |g_i(x)| \ \forall i = 1 \dots n\}$ with f_i and $g_i \in \mathcal{A}$, and $\bowtie_i \in \{\leq, <\}$ will be called **basic semianalytic**.

Remark 1.1.2. With a repeated use of the rule $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ one can show that any semianalytic subset of X is a finite union of basic semianalytic subsets.

Definition 1.1.3. Let (X, S) be a *k-germ* in the sense of [Ber93, 3.4]; this just means that S is a subset of X. An **constructible!elementary constructible datum** of (X, S), is the following datum. Let $f, g \in \mathcal{A}$. Let also r and s be two real numbers such that $r > s > 0$ and $s \in \sqrt{|k^*|}$. Let

$$Y = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f - tg)) \xrightarrow{\varphi} \mathcal{M}(\mathcal{A}) = X$$

and let $R \subseteq Y$ be a semianalytic subset of Y. Let us set

$$T := \varphi^{-1}(S) \cap \{y \in R \mid g(y) \neq 0 \text{ and } |f(y)| \leq s|g(y)|\}.$$

Then $(Y, T) \xrightarrow{\varphi} (X, S)$ is an elementary constructible datum. If $\psi : (Y', T') \simeq (Y, T)$ is an isomorphism of *k-germs*, and $(Y, T) \xrightarrow{\varphi} (X, S)$ is an elementary constructible datum, if we set $\varphi' = \varphi \circ \psi$, then we will also say that $(Y', T') \xrightarrow{\varphi'} (X, S)$ is an elementary constructible datum.

1. However, it has to be noted that we could have written this proof in the context of adic spaces, and used a similar argument of quasi-compactness.

Remark 1.1.4. If $(Y,T) \xrightarrow{\varphi} (X,S)$ is an elementary constructible datum, then $\varphi(T) \subset S$, and φ realizes a homeomorphism between T and its image $\varphi(T)$. Moreover

$$\{y \in Y \mid |f(y)| \leq s|g(y)| \neq 0\}$$

is an analytic domain of Y, and can be identified through φ with the analytic domain of X

$$\{x \in X \mid |f(x)| \leq s|g(x)| \neq 0\}.$$

Definition 1.1.5. Let (X,S) be a k-germ. A **constructible datum** of (X,S) is a sequence

$$(Y,T) = (X_n,S_n) \xrightarrow{\varphi_n} (X_{n-1},S_{n-1}) \to \cdots \to (X_1,S_1) \xrightarrow{\varphi_1} (X_0,S_0) = (X,S)$$

where for $i = 1 \dots n$, $(X_i,S_i) \xrightarrow{\varphi_i} (X_{i-1},S_{i-1})$ is an elementary constructible datum. Let $\varphi = \varphi_1 \circ \ldots \circ \varphi_n$. Then we will denote this constructible datum by

$$(Y,T) \overset{\varphi}{\dashrightarrow} (X,S).$$

We will say that the complexity of φ is n.

In the particular case $S = X$, i.e. $(X,S) = (X,X)$, we will denote the constructible datum by:

$$(Y,T) \overset{\varphi}{\dashrightarrow} X,$$

and we will call it a constructible datum of (X,S). This is actually the case that will mainly interest us, but partly for technical reasons we have chosen to use *k-germs*.

Remark 1.1.6. If $(Y,T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum, it follows easily from the above definitions that T is a semianalytic subset of Y.

Remark 1.1.4 implies that if $(Y,T) \overset{\varphi}{\dashrightarrow} (X,S)$ is a constructible datum, $\varphi_{|T} : T \xrightarrow{\varphi_{|T}} S$ induces a homeomorphism between T and $\varphi(T)$. It is also clear that if $(Z,U) \overset{\psi}{\dashrightarrow} (Y,T)$ is a constructible datum and $(Y,T) \overset{\varphi}{\dashrightarrow} (X,S)$ is another one, then $(Z,U) \overset{\varphi\circ\psi}{\dashrightarrow} (X,S)$ is also a constructible datum.

We want to point out that in the definition of a constructible datum, n cannot be recovered from φ alone.

Definition 1.1.7. Let $(X_i,S_i) \overset{\varphi_i}{\dashrightarrow} (X,S)$, $i = 1 \dots m$ be m constructible data of the k-germ (X,S). We will say it forms a **constructible cover** of (X,S) if $\bigcup_{i=1}^{m} \varphi_i(S_i) = S$.

Definition 1.1.8. Let X be a k-affinoid space. A subset C of X is said to be an **overconvergent constructible subset** of X if there exist m constructible data $(X_i,S_i) \overset{\varphi_i}{\dashrightarrow} X$ for $i = 1 \dots m$ such that $\bigcup_{i=1}^{m} \varphi_i(S_i) = C$.

Remark 1.1.9. Using the notations of definition 1.1.3, when $(Y,T) \xrightarrow{\varphi} (X,S)$ is an elementary constructible datum, with $Y = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f-tg))$, then T (and hence $\varphi(T)$) are defined with the function t which mimics the function $\dfrac{f}{g}$, when it has a sense and its norm is $\leq s$. In addition the condition $r > s$ is here to make sure that the new functions of $\mathcal{B}$ are overconvergent in $t = \dfrac{f}{g}$, that we see as a function on the analytic domain $\{x \in X \mid |f(x)| \leq s|g(x)| \neq 0\}$.

The following three results are formal consequences of the previous definitions.

Lemma 1.1.10. *If* $(Y,T) \xrightarrow{\varphi} (X,S)$ *is an elementary constructible datum and* $(Z,U) \xrightarrow{\varphi} (X,S)$ *is a morphism of* k*-germs, let us consider the cartesian product of* k*-germs:*

$$\begin{array}{ccc} (Y,T) & \xrightarrow{\varphi} & (X,S) \\ {\scriptstyle \psi'}\uparrow & & \uparrow{\scriptstyle \psi} \\ (Y,T)\times_{(X,S)}(Z,U) & \xrightarrow{\varphi'} & (Z,U) \end{array}$$

Then, $(Y,T)\times_{(X,S)}(Z,U) \xrightarrow{\varphi'} (Z,U)$ *is an elementary constructible datum. Moreover if we set*

$$(Y,T)\times_{(X,S)}(Z,U) = (Y',T')$$

then, $(\varphi\circ\psi')(T') = \varphi(T)\cap\psi(U)$.

Corollary 1.1.11. *Let* $(Y,T) \overset{\varphi}{\dashrightarrow} (X,S)$ *be a constructible datum*

$$(Y,T) = (X_n,S_n) \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_1} (X_0,S_0) = (X,S)$$

and let $(X',S') \xrightarrow{\psi} (X,S)$ *be a morphism of* k*-germs. Let us consider the cartesian product :*

$$\begin{array}{ccc} (Y,T) & \overset{\varphi}{\dashrightarrow} & (X,S) \\ {\scriptstyle \psi'}\uparrow & & \uparrow{\scriptstyle \psi} \\ (Y',T') & \overset{\varphi'}{\dashrightarrow} & (X',S') \end{array}$$

Then $(Y',T') \overset{\varphi'}{\dashrightarrow} (X',S')$ *is a constructible datum and* $(\psi\circ\varphi')(T') = \varphi(T)\cap\psi(S')$.

Corollary 1.1.12. *Let* $(X_1,T_1) \overset{\varphi}{\dashrightarrow} (X,S)$ *and* $(X_2,T_2) \overset{\psi}{\dashrightarrow} (X,S)$ *be two constructible data (with the same target). Let us consider the fibered product*

$$\begin{array}{ccc} (X_1,T_1) & \overset{\varphi}{\dashrightarrow} & (X,S) \\ {\scriptstyle \psi'}\,\vdots & & \vdots\,{\scriptstyle \psi} \\ (Z,U) & \overset{\varphi'}{\dashrightarrow} & (X_2,T_2) \end{array}$$

Then $(Z,U) \overset{\psi'}{\dashrightarrow} (X_1,T_1)$ *and* $(Z,U) \overset{\varphi'}{\dashrightarrow} (Y_2,T_2)$ *are constructible data. Moreover* $(\varphi\circ\psi')(U) = (\psi\circ\varphi')(U) = \varphi(T_1)\cap\psi(T_2)$.

Proof. Lemma 1.1.10 is a direct consequence of definition 1.1.3. Corollary 1.1.11 is then proved by induction on the complexity of φ using lemma 1.1.10. Similarly, corollary 1.1.12 is proved by induction on the complexity of ψ using corollary 1.1.11. □

Proposition 1.1.13.

1. *If* $T\subseteq X$ *is a semianalytic subset of* X *then* T *is an overconvergent constructible subset of* X.
2. *Let* $C\subseteq T$ *be an overconvergent constructible subset of* Y *and let* $(Y,T) \overset{\varphi}{\dashrightarrow} X$ *be a constructible datum. Then* $\varphi(C)$ *is on overconvergent constructible subset of* X.
3. *The class of overconvergent constructible subsets of* X *is stable under finite boolean combinations.*

Proof.

1. Consider the elementary constructible datum $(X,T) \xrightarrow{id} X$.
2. By definition, there exist some constructible data $(Y_i,T_i) \overset{\varphi_i}{\dashrightarrow} Y$, for $i = 1 \ldots m$, such that $C = \bigcup_{i=1}^{m} \varphi_i(T_i)$. Now if we define $\psi_i := \varphi \circ \varphi_i$, then $(Y_i,T_i) \overset{\psi_i}{\dashrightarrow} X$ are m constructible data, and $\varphi(C) = \varphi(\bigcup_{i=1}^{m} \varphi_i(Y_i)) = \bigcup_{i=1}^{m} \psi_i(T_i)$, so it is an overconvergent constructible subset of (X,S).
3. Stability under finite union is a direct consequence of the definition 1.1.8, as for intersection, it is a consequence of corollary 1.1.12. And if $C \subseteq X$ is an overconvergent constructible subset of X, let us show that $X \setminus C$ is also overconvergent constructible. By definition, $C = \bigcup_{i=1}^{m} \varphi_i(S_i)$ where $(X_i,S_i) \overset{\varphi_i}{\dashrightarrow} X$ are some constructible data. We do it by induction on c, the maximum of the complexity of the φ_i's.
   If $c = 0$, then C is a semianalytic subset of X so $X \setminus C$ is semianalytic, hence overconvergent constructible.
   If $c > 0$ and we assume the result for $c' < c$, then
   $$X \setminus C = X \setminus (\bigcup_{i=1}^{m} \varphi_i(S_i)) = \bigcap_{i=1}^{m} (X \setminus \varphi_i(S_i))$$
   so we can assume that $m = 1$, that is to say, we can assume that $C = \varphi(T)$ where $(Y,T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum of complexity c. Then
   $$\varphi = \psi \circ \varphi' : (Y,T) \overset{\varphi'}{\dashrightarrow} (Y',T') \xrightarrow{\psi} X$$
   where the complexity of φ' is $c-1$ and ψ is an elementary constructible datum. Now
   $$X \setminus \varphi(T) = \psi(T' \setminus \varphi'(T)) \cup (X \setminus \psi(T'))$$
   because $\varphi'_{|T}$ and $\psi_{|T'}$ are injective maps. By induction hypothesis,
   $$T' \setminus \varphi'(T) = T' \cap (Y' \setminus \varphi'(T))$$
   is an overconvergent constructible subset of Y', thus according to (1), so is $\psi(T' \setminus \varphi'(T))$.
   Finally, if the elementary constructible datum ψ is associated with f, g, r and s, by definition,
   $$T' = \{y \in R \mid |f(y)| \leq s|g(y)| \neq 0\}$$
   for some semianalytic subset R of Y'. And if we define
   $$\tilde{T} = \{y \in Y' \setminus R \mid |f(y)| \leq s|g(y)| \neq 0\},$$
   then
   $$X \setminus \psi(T') = \psi(\tilde{T}) \cup \{y \in X \mid |f(y)| > s|g(y)|\} \cup \{y \in X \mid g(y) = 0\}.$$
   Thus, it is also overconvergent constructible in X.

□

Let $x \in X$, and U be an affinoid neighbourhood of x. Shrinking U if necessary, we can assume [Ber90, 2.5.15] that U is a rational domain of the form $X(\underline{r}^{-1}\frac{f}{g}) = \{p \in X \mid |f_i(x)| \leq r_i|g(x)|\}$ such that $X\left((\frac{r}{2})^{-1}\frac{f}{g}\right)$ still contains x. For each i, we pick a real number s_i such that $\frac{r_i}{2} < s_i < r_i$ and $s_i \in \sqrt{|k^\times|}$. For each i, we consider the elementary constructible datum $(X_i, S_i) \xrightarrow{\varphi_i} X$ defined by $X_i = \mathcal{A}\{r_i^{-1}t_i\}/(f_i - t_i g)$, and $S_i = \{p \in X_i \mid |f_i(p)| \leq s_i|g(p)| \text{ and } g(p) \neq 0\}$. One checks that $\varphi_i(S_i)$ is a neighbourhood of x. Now if we take the fibered product of all these elementary constructible data, we obtain (using corollary 1.1.12) the following constructible datum:

$$\left(X\left(\underline{r}^{-1}\frac{f}{g}\right), X\left(\underline{s}^{-1}\frac{f}{g}\right)\right) \overset{\varphi}{\dashrightarrow} X$$

Here φ just corresponds to the embedding of the affinoid domain $X(\underline{r}^{-1}\frac{f}{g})$. Moreover $\varphi\left(X(\underline{s}^{-1}\frac{f}{g})\right)$, that we might identify with $X\left(\underline{s}^{-1}\frac{f}{g}\right)$, is a neighbourhood of x. We can sum up this in the following lemma:

Lemma 1.1.14. *Let X be a strictly k-affinoid space. Let $x \in X$ and U be an affinoid neighbourhood of x. Then there exists a constructible datum $(Y, T) \overset{\varphi}{\dashrightarrow} X$ such that T is an affinoid domain of Y, φ is the embedding of an affinoid domain $Y \to X$ such that Y is in fact an affinoid subdomain of U, and $\varphi(T)$ is an affinoid neighbourhood of x.*

Corollary 1.1.15. *Let X be a strictly k-affinoid space. Being overconvergent constructible in X is a local property.*

Proof. First, if $S \subset X$ is overconvergent constructible, and U is an affinoid domain of X, then $S \cap U$ is overconvergent constructible.

On the other hand, let us assume that locally for the Berkovich topology, S is overconvergent constructible, that is to say, let us assume that for all $x \in X$ there exists an affinoid neighbourhood U of x such that $S \cap U$ is overconvergent constructible. Then according to lemma 1.1.14, there exists a constructible datum $(Y, T) \overset{\varphi}{\dashrightarrow} X$ such that $Y \xrightarrow{\varphi} X$ is the embedding of an affinoid domain, $Y \subset U$, and T is an affinoid neighbourhood of x. Then, since $T \subset U$, $\varphi^{-1}(S) \cap T$ is overconvergent constructible in T, and then $\varphi(T) \cap S$ is overconvergent constructible in X (see proposition 1.1.13 (2)). But since $\varphi(T)$ is an affinoid neighbourhood of x, by compactness of X we conclude that S is overconvergent constructible. □

1.1.2 Overconvergent subanalytic sets

We will denote by $\mathbb{B}$ (resp. $\mathbb{B}_r$ for $r > 0$) the closed disc of radius 1 (resp. r), and if n is an integer, $\mathbb{B}^n$ and $\mathbb{B}^n_r$ will denote the corresponding closed polydiscs.

More generally, if $\underline{r} = (r_1, \ldots, r_n) \in {\mathbb{R}_+^*}^n$ is a polyradius, we will denote by

$$\mathbb{B}_{\underline{r}} = \mathcal{M}(k\{\underline{r}^{-1}t\}) = \mathcal{M}(k\{r_1^{-1}t_1, \ldots, r_n^{-1}t_n\})$$

the polydisc of radius $\underline{r}$, and $\mathring{\mathbb{B}}(\underline{r})$ the corresponding open polydisc. When the number n will be clear from the context, we will write $\underline{1}$ for $(1, \ldots, 1) \in \mathbb{R}^n$, and $\underline{0}$ or 0 for $(0, \ldots, 0) \in \mathbb{R}^n$. Finally, $\underline{\rho} > \underline{r}$ will mean that $\rho_i > r_i$ for $i = 1 \ldots n$.

Definition 1.1.16. Let X be a strictly k-affinoid space. A subset $S \subset X$ is said to be an **overconvergent subanalytic set** of X if there exist $n \in \mathbb{N}$, $r > 1$, and $T \subseteq X \times \mathbb{B}^n_r$ a semianalytic subset such that $S = \pi(T \cap (X \times \mathbb{B}^n))$ where $\pi : X \times \mathbb{B}^n_r \to X$ is the natural projection.

Lemma 1.1.17. *Let $f : Y \to X$ be a morphism of strictly k-affinoid spaces and S an overconvergent subanalytic subset of X. Then $f^{-1}(S)$ is an overconvergent subanalytic subset of X. In particular, if V is a strictly affinoid domain of X and S an overconvergent subanalytic subset of X, then $S \cap V$ is an overconvergent subanalytic subset of V.*

Proof. Let $r > 1$ and $T \subseteq X \times \mathbb{B}^n_r$ be a semianalytic subset such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. Let us consider the following cartesian diagram:

$$\begin{array}{ccc} Y \times \mathbb{B}^n_r & \xrightarrow{f'} & X \times \mathbb{B}^n_r \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array} \tag{1.1}$$

Then $f^{-1}(S) = f^{-1}(\pi(T \cap (X \times \mathbb{B}^n))) = \pi'(f'^{-1}(T \cap (X \times \mathbb{B}^n)))$. The last inequality holds because (1.1) is a cartesian diagram. Now $\pi'(f'^{-1}(T \cap (X \times \mathbb{B}^n))) = \pi'(f'^{-1}(T) \cap (Y \times \mathbb{B}^n)) = \pi'^{-1}(T' \cap (Y \times \mathbb{B}^n))$ where $T' = f'^{-1}(T)$ is a semianalytic subset of $Y \times \mathbb{B}^n_r$. Hence $f^{-1}(S) = \pi'(T' \cap (Y \times \mathbb{B}^n))$ is an overconvergent subanalytic subset of Y. □

Lemma 1.1.18. *Let $\varphi : X \to Y$ be a closed immersion.*

1. *If φ is a morphism of affinoid spaces, and S is a semianalytic subset of X, then $\varphi(S)$ is a semianalytic subset of Y.*
2. *Let S be an overconvergent subanalytic subset of X, then $\varphi(S)$ is an overconvergent subanalytic subset of Y.*

Proof.

1. Write $Y = \mathcal{M}(\mathcal{A})$ and $X = \mathcal{M}(\mathcal{A}/\mathcal{I})$ where $\mathcal{I} = (a_1, \dots, a_m)$ is an ideal of $\mathcal{A}$. Then, if $S = \{x \in X \mid |f_i(x)| \bowtie_i |g_i(x)|,\ i = 1 \dots n\}$ with $f_i, g_i \in \mathcal{A}/\mathcal{I}$, we can find functions $F_i, G_i \in \mathcal{A}$ such that $\overline{F_i} = f_i$ and $\overline{G_i} = g_i$. In that case one checks that,
$$\varphi(S) = \{y \in Y \mid |F_i(y)| \bowtie_i |G_i(y)|,\ i = 1 \dots n\} \cap \{y \in Y \mid a_j(y) = 0,\ j = 1 \dots m\},$$
which is indeed semianalytic.
2. Since the problem is local on Y, we can assume that Y is affinoid, and hence, X is also affinoid. By definition, this means that there exists $T \subseteq X \times \mathbb{B}^n_r$ for some $r > 1$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. We then consider the following cartesian diagram:
$$\begin{array}{ccc} X \times \mathbb{B}^n_r & \xrightarrow{\varphi'} & Y \times \mathbb{B}^n_r \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\varphi} & Y \end{array}$$
But φ' is also a closed immersion, so according to (1), $T' = \varphi'(T)$ is a semianalytic subset of $Y \times \mathbb{B}^n_r$. Then one checks that
$$\pi\left(T' \cap (Y \times \mathbb{B}^n)\right) = \pi\left(\varphi'(T) \cap (Y \times \mathbb{B}^n)\right) = \pi\left(\varphi'(T \cap (X \times \mathbb{B}^n)\right) = \varphi(\pi'(T \cap (X \times \mathbb{B}^n)) = \varphi(S).$$

□

Lemma 1.1.19. *Let us assume that $\underline{s} \in \sqrt{|k^\times|}^n$. Then, $k\{\underline{s}^{-1}X\}$ is a strictly k-affinoid algebra (see [Ber90, 2.1.1] and [BGR84, 6.1.5.4]). For the same reasons, if $\underline{r} > \underline{s}$, and $T \subseteq X \times \mathbb{B}_{\underline{r}}$ is a semianalytic subset, then $\pi\left(T \cap (X \times \mathbb{B}_{\underline{s}})\right)$ is an overconvergent subanalytic subset of X.*

Proof. Indeed let $\underline{s} \in \sqrt{|k^\times|}^n$ and $\underline{r} \in \mathbb{R}^n$ such that $\underline{s} < \underline{r}$, and $T \subseteq X \times \mathbb{B}_{\underline{r}}$ a semianalytic subset of $X \times \mathbb{B}_{\underline{r}}$. Let us show that $\pi(T \cap (X \times \mathbb{B}_{\underline{s}})$ is overconvergent subanalytic in the sense of definition 1.1.16. To avoid complications, we assume that $n = 1$ (but the proof is similar for an arbitrary n). Let then $s \in \sqrt{|k^\times|}$ and $r > s$. Up to multiplication by some $\mu \in k^\times$ small enough, we can assume that $s \leq 1$. Since $s \in \sqrt{|k^\times|}$, there exist $\lambda \in k^\times$ and $m \in \mathbb{N}$ such that $s^m = |\lambda|$. Then in

$$\mathbb{B}_{(r,(\frac{r}{s})^m)} = \mathcal{M}(k\{r^{-1}y, \left(\left(\frac{r}{s}\right)^m\right)^{-1} t\})$$

let us consider the Zariski-closed subset defined by $y^m = \lambda t$, i.e. $V(y^m - \lambda t)$. Then, the map:

$$\begin{array}{ccc} \mathbb{B}_r & \to & \mathbb{B}_{(r,(\frac{r}{s})^m)} \\ x & \mapsto & (x, \frac{x^m}{\lambda}) \end{array}$$

identifies $\mathbb{B}_r$ with the Zariski closed subset $V(y^m - \lambda t)$ and moreover, since $s \leq 1$

$$\begin{array}{ccc} \mathbb{B}_s & \to & \mathbb{B}^2 \\ x & \mapsto & (x, \frac{x^m}{\lambda}) \end{array}$$

identifies $\mathbb{B}_s$ with the Zariski-closed subset of $\mathbb{B}^2$, $V(y^m - \lambda t)$. Taking the fibre product with X we then obtain:

$$\begin{array}{ccccc} X \times \mathbb{B}_r & \xrightarrow{\simeq} & V(y^m - \lambda t) & \overset{\alpha}{\hookrightarrow} & X \times \mathbb{B}_{(r,(\frac{r}{s})^m)} \\ \uparrow & & \uparrow & & \uparrow \\ X \times \mathbb{B}_s & \xrightarrow{\simeq} & V(y^m - \lambda t) & \overset{\beta}{\hookrightarrow} & X \times \mathbb{B}^2 \\ & \searrow^{\pi} & & \swarrow_{\pi} & \\ & & X & & \end{array}$$

Hence if $T \subseteq X \times \mathbb{B}_r$ is semianalytic, $T' := \alpha(T)$ is also semianalytic in $X \times \mathbb{B}_{(r,(\frac{r}{s})^m)}$ and $\alpha(T) \cap (X \times \mathbb{B}^2) = \beta(T \cap (X \times \mathbb{B}_s))$. So $\pi(T \cap (X \times \mathbb{B}_s)) = \pi(T' \cap (X \times \mathbb{B}^2)$ is well overconvergent subanalytic in the sense of definition 1.1.16. □

We now take a few lines to explain an alternative definition which emphasizes the relevance of *k-𝒢erms* in this context.

Definition 1.1.20. Let (X, S) be a *k-𝒢erm* [Ber93, 3.4]. A subset $T \subseteq S$ is a semianalytic subset of (X, S) if there exists a representative (W, S) of (X, S), with W a k-affinoid space, and $R \subseteq W$ a semianalytic subset of W such that $T = S \cap R$.

Lemma 1.1.21. *Let $\mathcal{W}$ be a neighbourhood of $X \times \mathbb{B}^n$ in $X \times \mathbb{A}_k^{n,an}$. Then there exists $r > 1$ such that $\mathcal{W} \supseteq X \times \mathbb{B}_r^n$.*

Proof. If necessary, we can assume that $\mathcal{W} \subseteq X \times \mathbb{B}_s^n$ with $s > 1$ and also assume that $\mathcal{W}$ is open. In that case, $Z := (X \times \mathbb{B}_s^n) \setminus \mathcal{W}$ is a compact subset of $X \times \mathbb{B}_s^n$, and by assumption $Z \cap (X \times \mathbb{B}^n) = \emptyset$, so

$$Z \subseteq \bigcup_{i=1}^{n} \bigcup_{r>1} \{p \in X \times \mathbb{B}_s^n \mid |T_i(p)| > r\}.$$

Hence by compactness, there exists $r > 1$ such that $Z \subseteq \bigcup_{i=1}^{n} \{p \in X \times \mathbb{B}_s^n \mid |T_i(p)| > r\}$, which says that $\mathcal{W} \supseteq X \times \mathbb{B}_r^n$. □

Hence S is an overconvergent subanalytic subset of X if and only if there exist an integer n, and T a semianalytic subset of the k-$\mathscr{G}erm$ $(X \times \mathbb{A}_k^{n,an}, X \times \mathbb{B}^n)$ such that $S = \pi(T)$.

1.1.3 Weierstrass preparation

In this section, A will be a (ultrametric) complete normed ring i.e. satisfies the inequality $\|ab\| \leq \|a\|\|b\|$ and $\|a+b\| \leq \max(\|a\|, \|b\|)$ [BGR84, 1.2.1.1].

If $r > 0$, on $A\{r^{-1}X\}$ we will consider the following norm: if $g = \sum_{n \in \mathbb{N}} a_n X^n \in A\{r^{-1}X\}$ then $\|g\| = \max_{n \geq 0} \|a_n\| r^n$.

If $m \in \mathbb{N}$, we will denote by $A_m[X]$ the subset of $A[X]$ made of the polynomials of degree less or equal to m.

Definition 1.1.22. An element $u \in A$ is a **multiplicative unit** of A if u is invertible and for all $a \in A$, $\|ua\| = \|u\|\|a\|$.

Note that if u and v are multiplicative units, so is uv.

Lemma 1.1.23. *An element $u \in A$ is a multiplicative unit if and only if $u \in A^*$ and $\|u^{-1}\| = \|u\|^{-1}$.*

Proof. If u is a multiplicative unit, $1 = \|uu^{-1}\| = \|u\|\|u^{-1}\|$, so $\|u^{-1}\| = \|u\|^{-1}$.

Conversely let us assume that u is invertible and that $\|u^{-1}\| = \|u\|^{-1}$. Let then $a \in A$. The following holds:

$$\|a\| = \|u^{-1}(ua)\| \leq \|u^{-1}\|\|ua\| = \|u\|^{-1}\|ua\|.$$

So $\|ua\| \geq \|u\|\|a\|$. Since in any case the reverse inequality $\|ua\| \leq \|u\|\|a\|$ holds, we conclude that $\|ua\| = \|u\|\|a\|$. ☐

Remark 1.1.24. As a consequence, if $u \in A$ and $\|u\| < 1$, then $(1+u)$ is a multiplicative unit because

$$\|1+u\| = 1 = \|\sum_{n \geq 0} (-u)^n\| = \|(1+u)^{-1}\|$$

Let us note also that if u is a multiplicative unit, for all $x \in \mathcal{M}(A)$, $|u(x)| = \|u\|$. Indeed, the definition of $\mathcal{M}(\mathcal{A})$ implies that

$$|u(x)| \leq \|u\|, \tag{1.2}$$

hence $1 = |u(x)||u^{-1}(x)| \leq \|u\|\|u^{-1}\| = 1$. So the inequality (1.2) could not be strict, thus $|u(x)| = \|u\|$.

Remark 1.1.25. If $\varphi : A \to B$ is a **contractive morphism** of normed rings (i.e. $\|\varphi(a)\| \leq \|a\|$ for all a in A), then φ sends multiplicative units to multiplicative units. Indeed we have the sequence of inequalities:

$$1 = \|\varphi(u)\varphi(u)^{-1}\| \leq \|\varphi(u)\|\|\varphi(u^{-1})\| \leq \|u\|\|u^{-1}\| = 1.$$

So there were only equalities and $\varphi(u)$ is a multiplicative unit because $\|\varphi(u)\| = \|u\|$, and $\|\varphi(u)^{-1}\| = \|u^{-1}\| = \|u\|^{-1} = \|\varphi(u)\|^{-1}$.

This remark will apply in the following context: when $\mathcal{A}$ is a strictly k-affinoid algebra and we look at a morphism $\varphi : \mathcal{A} \to \mathcal{B} = \mathcal{A}\{r^{-1}X\}/I$ with I any ideal, and $\mathcal{B}$ is equipped with the quotient norm inherited from $\mathcal{A}\{r^{-1}X\}$. In this situation, φ is contractive. This is the case when we consider φ the morphism of a constructible datum $(Y,T) \overset{\varphi}{\dashrightarrow} X$.

Note that if φ is not contractive, multiplicative units are not necessarily preserved. For instance consider $\mathcal{A} = k\{t\}$ and $\mathcal{B} = k\{2^{-1}x, y\}/(y - x^2)$ that we equip with the residue norm. These k-affinoid algebras are isomorphic through $\varphi : t \mapsto x$, and if we choose $\pi \in k$ such that $\frac{1}{2} < |\pi| < 1$, then $u := 1 + \pi t$ is a multiplicative unit of $\mathcal{A}$, but not $\varphi(u)$. Note however that if the field k is stable (for instance in our situation, where k is a non-Archimedean complete field, k is stable if $char(\tilde{k}) = 0$, or if it is algebraically closed, or a discrete valuation field [BGR84, 3.6.2]), one might say that any morphism of reduced affinoid algebra is contractive. Indeed, if k is stable, and $\mathcal{A}$ is a reduced affinoid algebra, then it is a distinguished affinoid algebra [BGR84, 6.4.3], i.e. the supremum seminorm is a residue norm on $\mathcal{A}$. If $\mathcal{B}$ is also reduced, so that the supremum seminorm is an admissible norm on it, and $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of affinoid algebras, then φ is contractive.

Definition 1.1.26. Let $r > 0$ be a real number and $s \in \mathbb{N}$. An element $g = \sum_{n\geq 0} g_n X^n$ of $A\{r^{-1}X\}$ is called **X-distinguished** of order s if g_s is a multiplicative unit, $\|g_s\|r^s = \|g\|$ and for all $n > s$, $\|g_n\|r^n < \|g_s\|r^s$. Note that in that case, g is necessarily a non zero element since $g_s \neq 0$.

Remark 1.1.27. We can extend the previous remark saying that if $\varphi : A \to B$ is a contractive morphism and $g = \sum_{n\in\mathbb{N}} g_n X^n \in A\{r^{-1}X\}$ is X-distinguished of order s, then $\varphi(g) = \sum_{n\in\mathbb{N}} \varphi(g_n) X^n \in B\{r^{-1}X\}$ and it is an X-distinguished element of $B\{r^{-1}X\}$ of order s. This applies in particular when φ is the morphism of a constructible datum $(Y,T) \overset{\varphi}{\dashrightarrow}$.

Lemma 1.1.28. *Let $g = \sum_{m\in\mathbb{N}} g_m X^m \in A\{r^{-1}X\}$ be X-distinguished of order s.*

1. *Then for all $q = \sum_{k\in\mathbb{N}} q_k X^k \in A\{r^{-1}X\}$, $\|gq\| = \|g\|\|q\|$.*
2. *Let us set $gq = \sum_{l\in\mathbb{N}} c_l X^l$, and let us assume that $q \neq 0$. Let us denote by k_0 the greatest rank such that $\|q_{k_0}\|r^{k_0} = \|q\|$. Then $\|gq\| = \|c_{s+k_0}\|r^{s+k_0}$ and $\|c_{s+k_0}\| = \|g_s\|\|q_{k_0}\|$.*

Proof. First, without any hypothesis, it is true that

$$|gq\| \leq \|g\|\|q\|. \tag{1.3}$$

Conversely, by definition,

$$c_{s+k_0} = \sum_{m+k=s+k_0} g_m q_k. \tag{1.4}$$

Let then m and k be two integers such that $m + k = s + k_0$.

If $k > k_0$, by definition of k_0, $\|q_k\|r^k < \|q_{k_0}\|r^{k_0}$. So, using that g_s is a multiplicative unit, we obtain:

$$\|g_m q_k\|r^{s+k_0} = \|g_m q_k\|r^{m+k} \leq \|g_m\|r^m\|q_k\|r^k < \|g_s\|r^s\|q_{k_0}\|r^{k_0} = \|g_s q_{k_0}\|r^{s+k_0}.$$

Thus,

$$\|g_m q_k\| < \|g_s q_{k_0}\|. \tag{1.5}$$

If $k < k_0$, then $m > s$, and since $\|g_m\|r^m < \|g_s\|r^s$ (because g is X-distinguished of order s), we obtain with the same reasoning, that

$$\|g_m q_k\| < \|g_s q_{k_0}\|. \tag{1.6}$$

Thus, (1.4)–(1.6) and the ultrametric inequality imply that $\|c_{s+k_0}\| = \|g_s q_{k_0}\|$. And since g_s is a multiplicative unit, $\|g_s q_{k_0}\| = \|g_s\|\|q_{k_0}\|$.

Finally we obtain that $\|gq\| \geq \|g_s\|r^s\|q_{k_0}\|r^{k_0} = \|q\|\|g\|$, which with (1.3) ends the proof. □

Proposition 1.1.29. ***Weierstrass Division.*** *Let $g \in A\{r^{-1}X\}$ be X-distinguished of order s. If $f = \sum_{n\in\mathbb{N}} f_n X^n \in A\{r^{-1}X\}$ there exists an unique couple $(q, R) \in A\{r^{-1}X\} \times A_{s-1}[X]$ such that*

$$f = gq + R. \tag{1.7}$$

Moreover

$$|f\| = \max(\|g\|\|q\|, \|R\|). \tag{1.8}$$

Proof. First, let us show that if a couple (q, R) satisfies (1.7), then it must satisfy the equality (1.8). Because of the ultrametric inequality, $\|f\| \leq \max(\|g\|\|q\|, \|R\|)$. For the reverse inequality, we distinguish two cases.

If $\|gq\| \neq \|R\|$, then $\|f\| = \max(\|gq\|, \|R\|) = \max(\|g\|\|q\|, R\|\|)$ according to lemma 1.1.28.

Otherwise $\|gq\| = \|g\|\|q\| = \|R\|$, and we use again lemma 1.1.28 and its notations (so $gq = \sum_{l\in\mathbb{N}} c_l X^l$). We get $\|gq\| = \|c_{s+k_0}\|r^{s+k_0}$. Since R is a polynomial of degree d with $d < s$, and since $f = gq + R$, and $d < s + k_0$, the coefficient f_{s+k_0} of f is c_{s+k_0}, hence $\|f\| \geq \|c_{s+k_0}\|r^{s+k_0} = \|g\|\|q\|$.

This finally proves that $\|f\| = \max(\|g\|\|q\|, \|R\|)$.

From this we can conclude that the couple (q, R) is unique because if $f = gq' + R'$ is another decomposition, we have $0 = g(q - q') + (R - R')$ and since $\|g\| \neq 0$, $\|q - q'\| = \|R - R'\| = 0$, i.e. $R = R'$ and $q = q'$.

Let us now show the existence of such a decomposition. Let us set

$$g' := \sum_{m=0}^{s} g_m X^m.$$

In particular, $\|g\| = \|g'\|$ because g is X-distinguished of degree s. Let us set

$$\kappa := \frac{\max_{m>s}(\|g_m\|r^m)}{\|g_s\|r^s} = \frac{\max_{m>s}(\|g_m\|r^m)}{\|g\|}.$$

Since g is X-distinguished of order s, $\kappa < 1$. Actually, if $\kappa = 0$ (which would mean that $g = g'$), replace κ by $\frac{1}{2}$. In any case $\|g - g'\| \leq \kappa\|g\|$ and $\kappa \in]0, 1[$.

Next, let $N \in \mathbb{N}$ and let us set

$$f' := \sum_{k=0}^{N} f_k X^k.$$

Let us assume that N is big enough to fulfil $\|f - f'\| \leq \kappa\|f\|$. In particular, $\|f'\| = \|f\|$.

By definition and hypothesis, $g' \in A[X]$ is of degree s and possesses an invertible dominant coefficient, which is g_s. Hence in $A[X]$, one can carry out euclidean division by g' [Lan02, 4.1.1], which gives $f' = g'q + R$, with $R \in A_{s-1}[X]$ and $q \in A[X]$. We can then apply the norm equality (1.8) that we have shown in the first part of the proof, (because g' is also X-distinguished of order s): $\|f'\| = \max(\|g'\|\|q\|, \|R\|)$. In particular $\|q\| \leq \frac{\|f'\|}{\|g'\|} = \frac{\|f\|}{\|g\|}$ so that

$$\|g\|\|q\| \leq \|f\|.$$

Moreover $\|R\| \leq \|f'\| = \|f\|$. Thus the following holds:

$$f = f' + (f - f') = g'q + R + (f - f') = gq + R + (f - f') + (g' - g)q.$$

By definition of g' and of κ, $\|g' - g\| \leq \kappa\|g\|$, so

$$\|(g' - g)q\| \leq \|g\|\|q\|\kappa \leq \kappa\|f\| \tag{1.9}$$

In addition, by hypothesis,

$$\|f - f'\| \leq \kappa\|f\|. \tag{1.10}$$

Hence if we set

$$h := f - f' + (g' - g)q = f - (gq + R),$$

according to (1.9) and (1.10), we obtain that $\|h\| \leq \kappa\|f\|$.

To sum up, we have found some $\kappa \in]0, 1[$ such that

$$\forall f \in A\{r^{-1}X\}, \text{ there exist } q' \in A\{r^{-1}X\}, R' \in A_{s-1}[X] \text{ such that } \|f-(gq'+R')\| \leq \kappa\|f\|. \tag{1.11}$$

This allows us to define by induction two Cauchy sequences $(q^i) \in A\{r^{-1}X\}$ and $(R^i) \in A_{s-1}[X]$ such that $\|f - (gq^i + R^i)\| \leq \kappa^i\|f\|$ in the following way.

We start with $(q^0, R^0) = (0, 0)$.

In order to perform the induction step, let $i > 0$ be given and let us assume that (q^i, R^i) is defined. We set $h^i := f - (gq^i + R^i)$, which by induction hypothesis fulfils $\|h^i\| \leq \kappa^i\|f\|$. According to (1.11), we can define $q' \in A\{r^{-1}X\}$ and $R' \in A_{s-1}[X]$ such that $h^i = gq' + R' + h'$ with $\|q'\| \leq \frac{\|h^i\|}{\|g\|} \leq \kappa^i\frac{\|f\|}{\|g\|}$, and $\|R'\| \leq \|h^i\| \leq \kappa^i\|f\|$ and $\|h'\| \leq \kappa\|h^i\| \leq \kappa^{i+1}\|f\|$. Then we set $q^{i+1} := q^i + q'$ and $R^{i+1} := R^i + R'$.

Then $\|f - (gq^{i+1} + R_{i+1})\| = \|h^i - (gq + R)\| = \|h'\| \leq \kappa^{i+1}\|f\|$. By construction $\|q^{i+1} - q^i\| = \|q'\| \leq \kappa^i\frac{\|f\|}{\|g\|}$ and $\|R^{i+1} - R^i\| = \|R'\| \leq \kappa^i\|f\|$, so these sequences are well Cauchy sequences. This ends our induction.

Now, by completeness of $A\{r^{-1}X\}$ and $A_{s-1}[X]$ the sequences (q^i) and (R^i) have a limit, that we denote by $q \in A\{r^{-1}X\}$ and $R \in A_{s-1}[X]$, which satisfy $f = gq + R$ as we wanted. □

Corollary 1.1.30. ***Weierstrass Preparation***. *Let $g \in A\{r^{-1}X\}$ be a X-distinguished element of order s. There exists an unique couple $(w, e) \in A_s[X] \times A\{r^{-1}X\}$ such that w is a unitary polynomial of degree s, e is a multiplicative unit of $A\{r^{-1}X\}$, and $g = ew$.*

Proof. Using Weierstrass division, we can write $X^s = gq+R$ with $\|X^s\| = \max(\|g\|\|q\|, \|R\|)$, and $R \in A[X]_{s-1}$. Let us set

$$w := X^s - R = gq.$$

So $w \in A_s[X]$ is a unitary polynomial. Since g is X-distinguished of order s, according to lemma 1.1.28, and if we denote by k_0 the greatest index such that $\|q_{k_0}\|r^{k_0} = \|q\|$, and $w = \sum_{l=0}^{s} w_l X^l$, we obtain

$$\|w\| = \|gq\| = \|(gq)_{s+k_0}\|r^{s+k_0} = \|w_{s+k_0}\|r^{s+k_0}.$$

But since $w \in A_s[X]$, necessarily, $s + k_0 = s$ and $k_0 = 0$. Hence, by definition of k_0, for all $k > 0$, $\|q_0\| > \|q_k\|r^k$.

The coefficient of degree s in gq being 1, (because $gq = X^s - R$), we have the equality

$$1 = g_0 q_s + g_1 q_{s-1} + \ldots + g_s q_0$$

and since $k_0 = 0$, and g is X-distinguished of order s, we obtain, with the same reasoning that we have already used in the course of the proof of lemma 1.1.28, that $\|g_s q_0\| > \|g_{s-i} q_i\|$ for $i = 1 \ldots s$. So $\|g_s q_0\| = \|1\| = 1$, and $g_s q_0 = 1 - (g_{s-1} q_1 + \ldots g_0 q_s)$, with $\|g_s q_1 + \ldots g_0 q_s\| < 1$. Thus, $g_s q_0$ is a multiplicative unit. Moreover, since g_s is also a multiplicative unit, q_0 is also a multiplicative unit, and $\|q_0\| = \|g_s\|^{-1}$. Hence

$$q = q_0\left(1 + \frac{q_1}{q_0} X + \ldots + \frac{q_k}{q_0} X^k + \ldots\right) \tag{1.12}$$

and since $k_0 = 0$ (so $\|q_i\|r^i < \|q_0\|$ for $i > 0$) and q_0 is a multiplicative unit, $\|\frac{q_i}{q_0}\|r^i < 1$ for all $i > 0$. Hence

$$1 + \frac{q_1}{q_0} X + \ldots + \frac{q_k}{q_0} X^k + \ldots$$

is a multiplicative unit of $A\{r^{-1}X\}$, and according to (1.12), q is also a multiplicative unit. So $g = q^{-1}(X^s - R)$, with q^{-1} a multiplicative unit and $X^s - R$ a unitary polynomial of degree s. So if we set $e := q^{-1}$, and $w = X^s - R$ we have the expected result: $g = ew$.

As for the uniqueness of this decomposition, if $g = ew$, e and w being as in the statement of the corollary, then $w = X^s + R$ with $R \in A_{s-1}[X]$, and $X^s = w - R = e^{-1}g + (-R)$ which is the Weierstrass division of X^s by g. Hence e and R are unique and w too because $w = X^s + R$. □

Let us assume that $\mathcal{A}$ is a k-affinoid algebra, let $(r_1, \ldots, r_n)$ be a polyradius, and let us set $A := \mathcal{A}\{r_1^{-1}X_1, \ldots, r_{n-1}^{-1}X_{n-1}\}$. Then if we set $r = r_n$, $\mathcal{A}\{r_1^{-1}X_1, \ldots, r_n^{-1}X_n\} = A\{r^{-1}X\}$, and we can introduce the notion of an element X-distinguished, apply Weierstrass theory to them, which corresponds to the classical one, especially if $\mathcal{A} = k$, where we find the classical Tate algebra $k\{r_1^{-1}X_1, \ldots, r_n^{-1}X_n\}$.

Now we state a result that we will need in the next section.

Lemma 1.1.31. *Let $\varepsilon > 0$ be given and $\underline{r} > 0$ be a polyradius. Let us assume that A is noetherian, and let us consider*

$$f = \sum_{\nu \in \mathbb{N}^n} f_\nu X^\nu \in A\{\underline{r}^{-1}X\}.$$

Then there exists a finite subset $J \subseteq \mathbb{N}^n$, and for all $\nu \in J$, a series $\phi_\nu \in A\{\underline{r}^{-1}X\}$ satisfying $\|\phi_\nu\| < \varepsilon$, such that

$$f = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu)$$

and such that in the ϕ_ν's, not terms X^μ with $\mu \in J$ appear. Moreover, if we fix some $\mu \in \mathbb{N}^n$, we can assume that $\mu \in J$.

Proof. Let us denote by $\mathcal{I}$ the ideal generated by the family $\{f_\nu\}_{\nu \in \mathbb{N}^n}$. Since A is noetherian, there exists J a finite subset of $\mathbb{N}^n$ such that $\mathcal{I} = A.(f_\nu)_{\nu \in J}$. So for all $\mu \in \mathbb{N}^n \setminus J$ one can find a decomposition $f_\mu = \sum_{\nu \in J} f_\nu a_\mu^\nu$ with $a_\mu^\nu \in A$. In fact, using [BGR84, 3.7.3], we can even assume[2] that there exists a real constant $C > 0$ such that

$$\forall \mu \in \mathbb{N}^n,\ \forall \nu \in J,\ \|a_\mu^\nu\| \leq C\|f_\mu\|. \tag{1.13}$$

Then, let us define for $\nu \in J$

$$\phi_\nu = \sum_{\mu \in \mathbb{N}^n \setminus J} a_\mu^\nu X^\mu.$$

Since $\|a_\mu^\nu\| \leq C\|f_\mu\|$, $\phi_\nu \in \mathcal{A}\{\underline{r}^{-1}X\}$. Hence, in $A\{\underline{r}^{-1}X\}$, the following equality is satisfied:

$$f = \sum_{\nu \in J} f_\nu (X^\nu + \sum_{\mu \mathbb{N}^n \setminus J} a_\mu^\nu X^\mu) = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu). \tag{1.14}$$

Now, if $\nu_0 \notin J$ we set $J' = J \cup \{\nu_0\}$, $\phi'_{\nu_0} := 0$, and for $\nu \in J$, $\phi'_\nu := \sum_{\mu \in \mathbb{N}^n \setminus J'} a_\mu^\nu X^\mu$. One checks that the properties mentioned above still hold, namely $\|a_\nu^\mu\| \leq C\|f_\mu\|$, where the constant C has not been changed, and

$$f = \sum_{\nu \in J'} f_\nu (X^\nu + \phi'_\nu).$$

Moreover,

$$C\|f_\mu\|\underline{r}^\mu \xrightarrow[\mu| \to +\infty]{} 0,$$

so there exists a finite set $K \subset \mathbb{N}^n$ such that

$$\forall \nu \in J,\ \forall \mu \in N^n \setminus K,\ \|a_\mu^\nu\| < \varepsilon.$$

Hence if we increase J adding the elements of $K \setminus J$ to J, we will manage to obtain a decomposition

$$f = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu)$$

such that $\|\phi_\nu\| < \varepsilon$ for all $\nu \in J$. □

1.1.4 Equivalence of the two notions

From now on, $\mathcal{A}$ will be a k-affinoid algebra, and $\underline{r} \in {\mathbb{R}_+^*}^n$ a polyradius such that $\underline{r} > \underline{1}$ and we will set $\mathcal{A}\{\underline{r}^{-1}X\} = \mathcal{A}\{r_1^{-1}X_1, \ldots, r_n^{-1}X_n\}$. If $\nu \in \mathbb{N}^n$ we will set

$$X^\nu := X_1^{\nu_1} X_2^{\nu_2} \ldots X_n^{\nu_n}.$$

If $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$, we will set

$$|\nu| = \max_{i=1\ldots n} \nu_i.$$

If $\underline{r} \in {\mathbb{R}_+^*}^n$ and $\nu \in \mathbb{N}^n$, we will set

$$\underline{r}^\nu = \prod_{i=1}^n r_i^{\nu_i}.$$

2. Indeed, consider

$$\begin{array}{cccc} \psi : & A^J & \to & \mathcal{I} \\ & (a_\nu)_{\nu \in J} & \to & \displaystyle\sum_{\nu \in J} a_\nu f_\nu \end{array}$$

According to [BGR84, 3.7.3.1], $\mathcal{I}$ is a complete normed A-module, and ψ is a continuous map of normed A-modules. Hence there exists a constant C such that $\|\psi(x)\| \leq C\|x\|$ for all $x \in A^J$.

When $\mu, \nu \in \mathbb{N}^n$, we will say that $\mu <_{lex} \nu$ when μ is smaller than ν with respect to the lexicographic order, that is to say when there exists an index m such that $\mu_m < \nu_m$ and $\mu_{m-1} = \nu_{m-1}, \dots, \mu_1 = \nu_1$.

If $\mathcal{A}$ is a k-affinoid algebra, $f = \sum_{\nu \in \mathbb{N}^n} a_\nu X^\nu \in \mathcal{A}\{\underline{r}^{-1}X\}$ and $x \in \mathcal{M}(\mathcal{A})$, we will denote by f_x the element of $\mathcal{H}(x)\{\underline{r}^{-1}X\}$ defined by

$$f_x = \sum_{\nu \in \mathbb{N}^n} a_\nu(x) X^\nu.$$

Since $\mathcal{A}$ is noetherian, we can apply lemma 1.1.31 to it.

Proposition 1.1.32. *Let $f = \sum_{\nu \in \mathbb{N}^n} f_\nu X^\nu \in \mathcal{A}\{\underline{r}^{-1}X\}$. There exists a constructible covering of X, $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X$, $i = 1..N$, such that, if we consider the following cartesian diagrams:*

$$\begin{array}{ccc} (X_i, S_i) & \overset{\varphi_i}{\dashrightarrow} & X \\ {\scriptstyle \pi_i} \uparrow & & \uparrow {\scriptstyle \pi} \\ X_i \times \mathbb{B}_{\underline{r}} & \overset{\varphi_i'}{\dashrightarrow} & X \times \mathbb{B}_{\underline{r}} \end{array}$$

and if we denote by $\mathcal{A}_i$ the k-affinoid algebra of X_i, for all $i = 1..N$, there exist $a_i \in \mathcal{A}_i$ and a function

$$g_i = \sum_{\nu \in \mathbb{N}^n} g_{i,\nu} X^\nu \in \mathcal{A}_i\{\underline{r}^{-1}X\}$$

such that

- *For all i, the family $\{g_{i,\nu}\}_{\nu \in \mathbb{N}^n}$ generates the unit ideal in $\mathcal{A}_i$.*
- *For all i, $\varphi_i'^*(f)_{|\pi_i^{-1}(S_i)} = (a_i g_i)_{|\pi_i^{-1}(S_i)}$.*

Proof. According to lemma 1.1.31, we can find a finite subset $J \subseteq \mathbb{N}^n$, and for $\nu \in J$ some $\phi_\nu \in \mathcal{A}\{\underline{r}^{-1}X\}$ with $\|\phi_\nu\| < 1$, such that:

$$f = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu).$$

Let us fix any $r > 1$, and for each $\nu \in J$, let us consider the constructible datum $(X_\nu, T_\nu) \overset{\varphi_\nu}{\dashrightarrow} X$ where the affinoid algebra of X_ν is $\mathcal{A}\{r^{-1}t_\mu\}_{\mu \in J \setminus \{\nu\}} / (f_\mu - t_\mu f_\nu)$, and

$$T_\nu := \{x \in X_\nu \mid \ |f_\kappa(x)| \leq |f_\nu(x)| \ \ \forall \kappa \in J \setminus \{\nu\} \text{ and } f_\nu(x) \neq 0\}.$$

This gives rise to the following cartesian diagrams:

$$\begin{array}{ccc} (X_\nu, T_\nu) & \overset{\varphi_\nu}{\dashrightarrow} & X \\ {\scriptstyle \pi'} \uparrow & & \uparrow {\scriptstyle \pi} \\ X_\nu \times \mathbb{B}_{\underline{r}} & \overset{\varphi_\nu'}{\dashrightarrow} & X \times \mathbb{B}_{\underline{r}} \end{array}$$

Now,

$$\varphi_\nu'^*(f) = f_\nu\big(X^\nu + \phi_\nu + \sum_{\mu \in J \setminus \{\nu\}} t_\mu (X^\mu + \phi_\mu)\big).$$

For $\nu \in J$, we set

$$g_\nu = X^\nu + \phi_\nu + \sum_{\mu \in J \setminus \{\nu\}} t_\mu (X^\mu + \phi_\mu).$$

Hence,

$$\varphi_\nu'^*(f) = f_\nu g_\nu.$$

Moreover, if we set $g_\nu = \sum_{\mu\in\mathbb{N}^n} g_{\nu,\mu}X^\mu$, according lemma 1.1.31, the coefficient of index ν, $g_{\nu,\nu}$, is 1, so the coefficients of g_ν generate the unit ideal. Finally, let us denote by $\mathcal{I}$ the ideal of $\mathcal{A}$ generated by the family $(f_\nu)_{\nu\in J}$. By construction, $\mathcal{I}$ also equals the ideal generated by $(f_\nu)_{\nu\in\mathbb{N}^n}$. Then, according to the definition of the $T_\nu' s$:

$$\bigcup_{\nu\in J} \phi_\nu(T_\nu) = \{x \in X \mid \exists \nu \in J \text{ such that } f_\nu(x) \neq 0\} = X \setminus V(\mathcal{I}).$$

Thus, if we set $T = V(\mathcal{I})$, then $(X,T) \xrightarrow{id} X$ is an elementary constructible datum and $id^*(f)_{|T} = f_{|T} = 0$.

Now if we regroup the constructible data $(X_\nu, T_\nu) \overset{\varphi_\nu}{\dashrightarrow} X$, for $\nu \in J$, with $(X,T) \xrightarrow{\varphi} X$, we obtain the desired constructible cover. □

Definition 1.1.33. Let $\underline{r} \in (\mathbb{R}_+^*)^n$ be a polyradius and $d_1, \dots, d_{n-1}$ some integers such that

$$\forall\, i = 1 \dots n-1,\ r_n^{d_i} \leq r_i. \tag{1.15}$$

Then

$$\sigma : \begin{cases} X_i \mapsto & X_i + X_n^{d_i} \qquad \text{for } 1 \leq i \leq n-1 \\ X_n \mapsto & X_n \end{cases} \tag{1.16}$$

is an automorphism of $\mathcal{A}\{\underline{r}^{-1}X\}$. We will call such an automorphism (as well as the automorphism it induces on the k-analytic space $\mathbb{B}_{\underline{r}}$) a **Weierstrass automorphism**.

Remark 1.1.34. If $\underline{r} > \underline{1}$, we will use that σ induces a 'classical' Weierstrass automorphism of $\mathcal{A}\{X_1, \dots, X_n\}$, hence of $X \times \mathbb{B}^n$.

Remind the following classical result. If $f \in k\{X_1, \dots, X_n\}$, then there exists a Weierstrass automorphism σ of $k\{X_1, \dots, X_n\}$ such that $\sigma(f)$ is X_n-distinguished. Roughly speaking, the next lemma says that if $\mathcal{A}$ is a k-affinoid algebra, $f \in \mathcal{A}\{X_1, \dots, X_n\}$ is overconvergent, then locally on $X = \mathcal{M}(\mathcal{A})$, we can obtain an analogous result.

Proposition 1.1.35. *Let $\mathcal{A}$ be a k-affinoid algebra. Let $X = \mathcal{M}(\mathcal{A})$ and let $x \in X$. Let $\underline{r} \in \mathbb{R}^n$ be a polyradius such that $\underline{r} > 1$.*

1. *Let $f \in \mathcal{A}\{\underline{r}^{-1}X\}$ such that $f_x \neq 0$. Then there exist an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x, a polyradius $\underline{\rho}$ such that $1 < \underline{\rho} \leq \underline{r}$, and σ a Weierstrass automorphism of $\mathcal{B}\{\underline{\rho}^{-1}X\}$ such that in $\mathcal{B}\{\underline{\rho}^{-1}X\}$*

$$\sigma(f) = ag$$

where $a \in \mathcal{B}$ and $g \in \mathcal{B}\{\underline{\rho}^{-1}X\}$ is X_n-distinguished.

2. *More generally, let us consider m functions $f_1, \dots, f_m \in \mathcal{A}\{\underline{r}^{-1}X\}$ such that for all i $(f_i)_x \neq 0$. Then there exist an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x, a polyradius $\underline{\rho}$ such that $1 < \underline{\rho} \leq \underline{r}$, and σ a Weierstrass automorphism of $\mathcal{B}\{\underline{\rho}^{-1}X\}$ such that for all i*

$$\sigma(f_i) = a_i g_i$$

where $a_i \in \mathcal{B}$ and $g_i \in \mathcal{B}\{\underline{\rho}^{-1}X\}$ is X_n-distinguished.

Proof. We first prove (1).

Step 1. Let us write

$$f = \sum_{\nu \in \mathbb{N}^n} f_\nu X^\nu \in \mathcal{A}\{\underline{r}^{-1}X\}.$$

Let us consider $\mu \in \mathbb{N}^n$ the greatest index with respect to the lexicographic order such that

$$\max_{\nu \in \mathbb{N}^n} |f_\nu(x)| = |f_\mu(x)|.$$

Since by assumption $f_x \neq 0$, it is true that $f_\mu(x) \neq 0$. According to lemma 1.1.31, there exists a finite set $J \subset \mathbb{N}^n$ such that $\mu \in J$, and for each $\nu \in J$ a series $\phi_\nu \in \mathcal{A}\{\underline{r}^{-1}X\}$ which satisfies $\|\phi_\nu\|_{\mathcal{A}\{\underline{r}^{-1}X\}} < 1$ such that

$$f = \sum_{\nu \in J} f_\nu (X^\nu + \phi_\nu). \tag{1.17}$$

Step 2. Let us consider some $\nu \in J$ and let us assume that

$$|f_\nu(x)| < |f_\mu(x)|.$$

Then we pick some $a, b \in \mathbb{R}$ such that

$$|f_\nu(x)| < a < b < |f_\mu(x)|.$$

Next, let us introduce the affinoid domain of X:

$$W := \{z \in X \mid |f_\nu(z)| \leq a \text{ and } b \leq |f_\mu(z)|\} = \mathcal{M}(\mathcal{B}).$$

By construction, W is an affinoid neighbourhood of x, f_μ is invertible in $\mathcal{B}$ and

$$\left\| \frac{f_\nu}{f_\mu} \right\|_{\mathcal{B}} \leq \frac{a}{b} < 1.$$

So we can write:

$$f_\nu(X^\nu + \phi_\nu) = f_\mu \left(\frac{f_\nu}{f_\mu}(X^\nu + \phi_\nu) \right).$$

Next we consider some polyradius $\underline{1} < \underline{\rho} \leq \underline{r}$. Clearly

$$\underline{\rho}^\nu \xrightarrow[\underline{\rho} \to 1]{} 1.$$

So we can chose some $\underline{\rho}$ close enough to $\underline{1}$ such that

$$\left\| \frac{f_\nu}{f_\mu} X^\nu \right\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < 1.$$

Since we already knew that $\|\phi_\nu\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < 1$ it follows that

$$\left\| \frac{f_\nu}{f_\mu}(X^\nu + \phi_\nu) \right\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < 1.$$

But since

$$f_\nu(X^\nu + \phi_\nu) = f_\mu \left(\frac{f_\nu}{f_\mu}(X^\nu + \phi_\nu) \right),$$

if we set

$$\phi'_\mu := \phi_\mu + \frac{f_\nu}{f_\mu}(X^\nu + \phi_\nu)$$

we still have that $\|\phi'_\mu\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < 1$ and

$$f_\mu(X^\mu + \phi_\mu) + f_\nu(X^\nu + \phi_\nu) = f_\mu(X^\mu + \phi'_\mu).$$

Hence we can remove ν from J and replace ϕ_μ by ϕ'_μ. The equality (1.17) will still be satisfied.

If we repeat this process for each $\nu \in J$ such that $|f_\nu(x)| < |f_\mu(x)|$, we can assume that

$$\forall \nu \in J,\ |f_\nu(x)| = |f_\mu(x)|.$$

Thus, according to the definition of μ, this implies that μ is the greatest index in J with respect to the lexicographic order.

Step 3. Then we set

$$d := 1 + \max_{\nu \in J} |\nu|.$$

Since by assumption $\underline{1} < \underline{r}$, if we take $s > 1$ which is close enough to 1, we can assert that

$$\underline{1} < (s^{d^{n-1}}, s^{d^{n-2}}, \dots, s^d, s) \le \underline{r}. \tag{1.18}$$

We fix a number $s > 1$, which satisfies (1.18), and we set

$$\underline{\rho} := (s^{d^{n-1}}, s^{d^{n-2}}, \dots, s^d, s). \tag{1.19}$$

In these conditions, it is easy to check that $\underline{\rho}$ satisfies condition (1.15) of definition 1.1.33 (actually, $\underline{\rho}$ has been defined in (1.19) to further this goal), so

$$\sigma : \begin{cases} X_1 & \mapsto & X_1 + X_n^{d^{n-1}} \\ \vdots & & \vdots \\ X_i & \mapsto & X_i + X_n^{d^{n-i}} \\ \vdots & & \vdots \\ X_{n-1} & \mapsto & X_{n-1} + X_n^d \\ X_n & \mapsto & X_n \end{cases}$$

defines a Weierstrass automorphism of $\mathcal{B}\{\underline{\rho}^{-1}X\}$. Then, for $\nu \in J \setminus \{\mu\}$

$$\sigma(f_\nu(X^\nu + \phi_\nu)) = f_\nu(\sigma(X^\nu) + \sigma(\phi_\nu)) = f_\mu\left(\frac{f_\nu}{f_\mu}(\sigma(X^\nu) + \sigma(\phi_\nu))\right).$$

Since $\|\sigma(\phi_\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} = \|\phi_\nu\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < 1$, we can chose s close enough to 1, so that

$$s\|\phi_\nu\| < 1. \tag{1.20}$$

Then we make the following calculation. If $\nu \in J$:

$$\|\sigma(X^\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} = \|X^\nu\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} = \prod_{k=1}^{n} \left(s^{d^{n-k}}\right)^{\nu_k} = s^{\left(\sum\limits_{k=1}^{n} \nu_k d^{n-k}\right)}. \tag{1.21}$$

Moreover, we remark that $\sum_{k=1}^{n} \nu_k d^{n-k}$ is nothing else but the integer encoded by ν in base d. Since by assumption, for all $\nu \in J \setminus \{\mu\}$ we have $\nu <_{lex} \mu$, it follows that for all $\nu \in J \setminus \{\mu\}$

$$\sum_{k=1}^{n} \nu_k d^{n-k} + 1 \leq \sum_{k=1}^{n} \mu_k d^{n-k}.$$

As a corollary,

$$s\|\sigma(X^\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} \leq \|\sigma(X^\mu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}}. \tag{1.22}$$

Let us now consider some $s' \in \mathbb{R}$, such that $1 < s' < s$ and let us consider

$$V := \{z \in X \mid \forall \nu \in J \setminus \{\mu\},\ |f_\nu(z)| \leq s'|f_\mu(z)|\}.$$

Then by construction, V is an affinoid neighbourhood of x. Let us then replace $\mathcal{B}$ by the affinoid algebra of V. Then by construction still, for all $\nu \in J \setminus \{\mu\}$,

$$\|\frac{f_\nu}{f_\mu}\|_{\mathcal{B}} \leq s' < s.$$

So according to (1.22)

$$\|\frac{f_\nu}{f_\mu}\sigma(X^\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < s\|\sigma(X^\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} \leq \|\sigma(X^\mu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}}.$$

So, according to (1.20), we can assume that

$$\|\frac{f_\nu}{f_\mu}\sigma(\phi_\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} \leq s'\|\sigma(\phi_\nu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} = s'\|\phi_\nu\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < 1 \leq \|\sigma(X^\nu)\|.$$

Thus

$$\sigma(f_\nu(X^\nu + \phi_\nu)) = f_\mu(\frac{f_\nu}{f_\mu}(\sigma(X^\nu) + \sigma(\phi_\nu)))$$

where

$$\|\frac{f_\nu}{f_\mu}(\sigma(X^\nu) + \sigma(\phi_\nu))\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < \|\sigma(X^\mu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}}.$$

Step 4. So

$$\sigma(f) = f_\mu\left(\sigma(X^\mu) + \sigma(\phi_\mu) + \sum_{\nu\in J\setminus\{\mu\}} \frac{f_\nu}{f_\mu}(\sigma(X^\nu) + \sigma(\phi_\nu))\right)$$

Hence if we set

$$\phi = \sigma(\phi_\mu) + \sum_{\nu\in J\setminus\{\mu\}} \frac{f_\nu}{f_\mu}(\sigma(X^\mu) + \sigma(\phi_\nu))$$

the preceding inequalities imply that $\|\phi\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}} < \|\sigma(X^\mu)\|_{\mathcal{B}\{\underline{\rho}^{-1}X\}}$, and by construction

$$\sigma(f) = f_\mu(\sigma(X^\mu) + \phi)).$$

It follows that $\sigma(X^\mu) + \phi$ is X_n-distinguished of order $\sum_{k=1}^{n} \mu_k d^{n-k}$, which ends the proof of (1).

For the proof of (2), it suffices to remark that we could have handled the proof of (1) simultaneously for all the $f_i's$. The main point being that in step 3, we have to take some d big enough that works for all $f_i's$ simultaneously. □

Remark 1.1.36. The main idea of the above proof is to take a function $f \in \mathcal{A}\{r^{-1}X_1, \dots, r^{-1}X_n\}$ for some $r > 1$ and to decrease the radius and apply a Weierstrass automorphism so that f becomes X_n-distinguished. To this end, in the proof, we introduce a polyradius $\underline{r} = (r_1, \dots, r_n)$ such that $\underline{1} < \underline{r} \leq (r, \dots, r)$, and one might think that this is an extra complication, and that it would have been sufficient to take some $1 < s \leq r$ with s sufficiently close to 1, and then to consider the polyradius $(s, \dots, s)$. But the problem is that with a polyradius such as $(s, \dots, s)$, the transformation σ of (1.16) (which should be our Weierstrass automorphism) would not be correctly defined.

Lemma 1.1.37. *If S is an overconvergent constructible subset of X, then S is an overconvergent subanalytic subset of X.*

Proof. It is sufficient to prove that if $(Y, T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum, then $\varphi(T)$ is overconvergent subanalytic in X.

We claim that if φ is a constructible datum of complexity n, there exist some polyradii $\underline{s}, \underline{r} \in \mathbb{R}^n$ such that $\underline{s} \in \sqrt{|k^\times|}^n$ and $0 < \underline{s} < \underline{r}$, and some closed immersion ι:

$$\begin{array}{ccc} Y & \overset{\iota}{\hookrightarrow} & X \times \mathbb{B}_{\underline{r}} \\ & \searrow^{\varphi} & \downarrow \pi \\ & & X \end{array}$$

such that $\iota(T) \subset X \times \mathbb{B}_{\underline{s}}$. Indeed this follows from the definition of a constructible datum, and is proved easily by induction on the complexity of the constructible datum φ.

Hence $\varphi(T) = \pi(\iota(T))$, and since $\iota(T)$ is a semianalytic subset of $X \times \mathbb{B}_{\underline{r}}$ such that

$$\iota(T) \subset X \times \mathbb{B}_{\underline{s}}$$

it follows from lemma 1.1.19 that $\pi(\iota(T))$ is an overconvergent subanalytic subset of X. □

Proposition 1.1.38. *Let $S \subset X$. If S is overconvergent subanalytic, S is also overconvergent constructible.*

Proof. Let S be an overconvergent subanalytic subset of X. By definition, there exist $r > 1$, T a semianalytic subset of $X \times \mathbb{B}^n_r$ such that $S = \pi\left(T \cap (X \times \mathbb{B}^n)\right)$. We then show by induction on n that S is overconvergent subanalytic.

If $n = 0$, there is nothing to prove since in that case, X is then a semianalytic subset of X, in particular it is an overconvergent constructible subset.

Let then $n > 0$ be given and let us assume that the theorem holds for integers $< n$. In order to prove the theorem, we can actually assume that T is a basic semianalytic subset (see remark 1.1.2), i.e. that there are $2m$ functions $f_1, \dots, f_m, g_1, \dots, g_m \in \mathcal{A}\{r^{-1}X_1, \dots, r^{-1}X_n\}$ and $\bowtie_j \in \{\leq, <\}$ for $j = 1 \dots m$ such that

$$T = \{x \in X \times \mathbb{B}^n_r \mid |f_j(x)| \bowtie_j |g_j(x)| \; j = 1 \dots m\}. \tag{1.23}$$

Step 1. According to proposition 1.1.32 we can find a constructible covering $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X$ where $X_i = \mathcal{M}(\mathcal{B}_i)$ which induces the following cartesian diagram

$$\begin{array}{ccc} X_i \times \mathbb{B}^n_r & \xrightarrow{\varphi'_i} & X \times \mathbb{B}^n_r \\ \pi_i \downarrow & & \downarrow \pi \\ X_i & \overset{\varphi_i}{\dashrightarrow} & X \end{array}$$

such that for all $j = 1 \dots m$,

$$\varphi'^{*}_{i}(f_j)_{|\pi_i^{-1}(S_i)} = (a^i_j F^i_j)_{|\pi_i^{-1}(S_i)} \tag{1.24}$$

$$\varphi'^{*}_{i}(g_j)_{|\pi_i^{-1}(S_i)} = (b^i_j G^i_j)_{|\pi_i^{-1}(S_i)} \tag{1.25}$$

where $a^i_j, b^i_j \in \mathcal{B}_i$, $F^i_j, G^i_j \in \mathcal{B}_i\{\underline{r}^{-1}X\}$, and the coefficients of F^i_j (resp. of G^i_j) generate the unit ideal in $\mathcal{B}_i$. Then for each i we set

$$T_i := \{x \in X_i \times \mathbb{B}^n_r \mid |a^i_j F^i_j(x)| \bowtie_j |b^i_j G^i_j(x)| \; j = 1 \dots m\}$$

So (1.24) and (1.25) imply precisely that

$$T_i \cap \pi_i^{-1}(S_i) = \varphi'^{-1}_{i}(T) \cap \pi_i^{-1}(S_i).$$

So if we set

$$U_i := \pi_i(T_i \cap (X_i \times \mathbb{B}^n))$$

then,

$$\varphi_i(S_i \cap U_i) = \varphi_i(S_i) \cap S$$

hence since the $\varphi_i(S_i)$ form a covering of X,

$$S = \bigcup_{i=1}^{n} \varphi(S_i \cap U_i).$$

So if we prove that $\varphi_i(S_i \cap U_i)$ is overconvergent constructible, we are done.

But actually, since each S_i is overconvergent constructible in X_i (it is even semianalytic, see remark 1.1.6) if we prove that U_i is an overconvergent constructible subset of X_i, then it will follow that $S_i \cap U_i$ is an overconvergent constructible subset of X_i, and then according to proposition 1.1.13 (2), $\varphi_i(S_i \cap U_i)$ will be overconvergent constructible in X. Thus, we restrict to prove that U_i is overconvergent constructible in X_i.

Step 2. We can then replace X by one of the X_i's and assume that T is defined by

$$T = \{x \in X \times \mathbb{B}^n_r \mid |a_j f_j(x)| \bowtie_j |b_j g_j(x)| \; j = 1 \dots m\} \tag{1.26}$$

with $a_j, b_j \in \mathcal{A}$, $f_j, g_j \in \mathcal{A}\{\underline{r}^{-1}X\}$ such that for all j, the coefficients of f_j (resp. of g_j) generate the unit ideal of $\mathcal{A}$. In this situation we must show that S is overconvergent constructible in X where

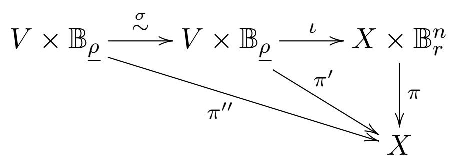
$$S = \pi(T \cap (X \times \mathbb{B}^n)).$$

Let then $x \in X$. The above property of the f_j's and $g_j's$ implies that $(f_j)_x \neq 0$ and $(g_j)_x \neq 0$. So we can apply proposition 1.1.35 to them. Thus there exist an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x, some polyradius $\underline{1} < \underline{\rho} \leq (r, \dots, r)$ and some Weierstrass automorphism σ of $\mathcal{B}\{\underline{\rho}^{-1}X\}$ such that for each j,

$$\sigma(f_j) = \alpha_j F_j \tag{1.27}$$

$$\sigma(g_j) = \beta_j G_j \tag{1.28}$$

where $\alpha_j, \beta_j \in \mathcal{B}$ and F_j, G_j are X_n-distinguished elements of $\mathcal{B}\{\underline{\rho}^{-1}X\}$. Let us then consider the following commutative diagram:



where ι is the embedding of the affinoid domain $V \times \mathbb{B}_{\underline{\rho}}$ in $X \times \mathbb{B}^n_{\underline{r}}$. Then let us set

$$T' := \iota^{-1}(T) \text{ and } T'' := \sigma^{-1}(\iota^{-1}(T)).$$

First it is clear that

$$\begin{aligned} S \cap V &= \pi(T \cap (X \times \mathbb{B}^n)) \\ &= \pi(T \cap (V \times \mathbb{B}^n)) \\ &= \pi'(T' \cap (V \times \mathbb{B}^n)) \\ &= \pi''(T'' \cap (V \times \mathbb{B}^n)) \end{aligned} \tag{1.29}$$

For the last equality (1.29), we use that the Weierstrass automorphism σ induces an isomorphism of $V \times \mathbb{B}^n$ as noticed in remark 1.1.34.

But since we know that being overconvergent constructible is a local property (see corollary 1.1.15), if we prove that $S \cap V$ is overconvergent constructible, then since x has been taken arbitrarily, and since V is an affinoid neighbourhood of x, this will conclude the proof. So we can restrict to prove that $\pi''(T'' \cap (V \times \mathbb{B}^n))$ is overconvergent constructible in V. Now according to (1.26)–(1.28), T'' is a semianalytic subset of $V \times \mathbb{B}_{\underline{\rho}}$ defined by inequalities between functions $a_j\alpha_j F_j$, $b_j\beta_j G_j$, where $a_j, \alpha_j, b_j, \beta_j \in \mathcal{B}$ and $F_j, G_j \in \mathcal{B}\{\underline{\rho}^{-1}X\}$ are X_n-distinguished.

Step 3. So replacing ρ by $\underline{r}$, X by V, T by T'', $a_j\alpha_j$ by a_j, $b_j\beta_j$ by b_j, F_j by f_j and G_j by g_j, we can assume that

$$T = \{x \in X \times \mathbb{B}_{\underline{r}} \mid |a_j f_j(x)| \bowtie_j |b_j g_j(x)| \ j = 1 \dots m\}. \tag{1.30}$$

where $a_j, b_j \in \mathcal{A}$ and $F_j, G_j \in \mathcal{A}\{\underline{r}^{-1}X\}$ are X_n distinguished in $\mathcal{A}\{\underline{r}^{-1}X\}$. Then, we apply the Weierstrass preparation theorem 1.1.30 to f_j and g_j. So there exist $e_j, e'_j \in \mathcal{A}\{\underline{r}^{-1}X\}$ some multiplicative units, and w_j, w'_j some unitary polynomials of $\mathcal{A}\{r_1^{-1}X_1, \dots, (r_{n-1})^{-1}X_{n-1}\}[X$ such that

$$\begin{aligned} f_j &= e_j w_j \\ g_j &= e'_j w'_j. \end{aligned}$$

So if we set

$$\begin{aligned} P_j &:= a_j w_j \\ Q_j &:= b_j w'_j, \end{aligned}$$

we have that $P_j, Q_j \in \mathcal{A}\{r_1^{-1}X_1, \dots, (r_{n-1})^{-1}X_{n-1}\}[X_n]$. In addition, since e_j, e'_j are multiplicative unit, for all $x \in X \times \mathbb{B}_{\underline{r}}$, $|e_j(x)| = \|e_j\| \in \sqrt{|k^\times|}$. So we finally obtain that

$$\begin{aligned} T &= \{x \in X \times \mathbb{B}_{\underline{r}} \mid |a_j f_j(x)| \bowtie_j |b_j g_j(x)| \ j = 1 \dots m\} \\ &= \{x \in X \times \mathbb{B}_{\underline{r}} \mid \|e_j\| |P_j(x)| \bowtie_j \|e'_j\| |Q_j(x)| \ j = 1 \dots m\}. \end{aligned} \tag{1.31}$$

Let us consider the projection along the last coordinate of $\mathbb{B}_{\underline{r}}$,

$$X \times \mathbb{B}_{\underline{r}} \xrightarrow{\pi_1} X \times \mathbb{B}_{(r_1, \dots, r_{n-1})} \xrightarrow{\pi_2} X$$

according to [Duc03, 2.5] $\pi_1(T \cap (X \times \mathbb{B}^n)$ is a semianalytic subset of $X \times \mathbb{B}_{(r_1, \dots, r_{n-1})}$. So by induction hypothesis,

$$\pi_2(\pi_1(T \cap (X \times \mathbb{B}^n))$$

is overconvergent constructible in X. Since $\pi_2 \circ \pi_1 = \pi$, this proves that S is overconvergent constructible and ends the proof. □

Remark 1.1.39.

We have then proved

Theorem 1.1.40. *Let X be a strictly k-affinoid space, and $S \subset X$. Then S is overconvergent subanalytic if and only if S is overconvergent constructible.*

Hence thanks to this theorem we can use some obvious properties of overconvergent subanalytic (resp. constructible) subsets to prove less obvious results about overconvergent constructible (resp. subanalytic) subsets. For instance we can obtain the non-trivial result concerning overconvergent subanalytic subsets:

Proposition 1.1.41. *Let X be a strictly k-affinoid space. The class of overconvergent subanalytic subsets of X is stable under finite boolean combination*[3].

Proof. This was proven for overconvergent constructible subsets in proposition1.1.13. □

In the same way, we obtain a non-obvious stability property for overconvergent constructible subsets:

Corollary 1.1.42. *Let $\underline{r} \in \mathbb{R}^n$ be a polyradius such that $\underline{r} > \underline{1}$, and $S \subseteq X \times \mathbb{B}_{\underline{r}}$ be an overconvergent subanalytic (or constructible) subset of $X \times \mathbb{B}_{\underline{r}}$. Then $\pi\left(S \cap (X \times \mathbb{B}^n)\right)$ is an overconvergent subanalytic (or constructible) subset of X.*

Proof. If S is an overconvergent subanalytic subset of $X \times \mathbb{B}_{\underline{r}}$, by definition, there exists $s > 1$, an integer m and T a semianalytic subset of $X \times \mathbb{B}_{\underline{r}} \times \mathbb{B}_s^m$ such that $S = \pi_2(T \cap ((X \times \mathbb{B}_{\underline{r}}) \times \mathbb{B}^m))$ where $\pi_2 : (X \times \mathbb{B}_{\underline{r}}) \times \mathbb{B}_s^m \to X \times \mathbb{B}_{\underline{r}}$ is the natural projection. Hence $\pi(S \cap (X \times \mathbb{B}^n)) = \pi_2(T \cap ((X \times \mathbb{B}^n) \times \mathbb{B}^m)) = \pi_2(T \cap (X \times \mathbb{B}^{n+m})$ where $\pi_2 : X \times \mathbb{B}_{\underline{r}} \times \mathbb{B}_s^m \to X$ is the natural projection (so $\pi_2 = \pi \circ \pi_1$). Hence S is an overconvergent subanalytic subset of X. □

1.1.5 From a global to a local definition

Definition 1.1.43. Let X be a good k-analytic space. A wide covering of X is a covering $\{X_i\}$ such that the $X_i's$ are affinoid domains of X and $\{\overset{\circ}{X_i}\}$ is also a covering of X.

Proposition 1.1.44. *Let X be a strictly k-affinoid space, and S a subset of X. The following assertions are equivalent:*

1. *S is an overconvergent subanalytic subset of X.*
2. *For all wide covering $\{X_i\}$ of X, $X_i \cap S$ is an overconvergent subanalytic subset of X_i.*
3. *There exists a wide covering $\{X_i\}$ of X such that $X_i \cap S$ is overconvergent subanalytic in X_i for all i.*
4. *For all $x \in X$ there exists an affinoid neighbourhood V of x such that $V \cap S$ is overconvergent subanalytic in V.*
5. *For all $x \in X$ there exist $V_1, \ldots V_n$ some affinoid domains of X such that $V_1 \cup \ldots \cup V_n$ is a neighbourhood of x and such that $V_i \cap S$ is overconvergent subanalytic in V_i for all i.*

The property (4) implies that the class of overconvergent subanalytic subsets is local in the sense of definition 1.0.1.

3. In fact, the only non-trivial result is that overconvergent subanalytic subsets are stable under taking complementary.

Proof. (1) $\Rightarrow$ (2) is obvious and is a consequence of lemma 1.1.17.

(2) $\Rightarrow$ (3) is clear.

(3) $\Leftrightarrow$ (4) follows from the compactness of X.

(4) $\Rightarrow$ (1) follows from the analogous statement for overconvergent constructible subsets (corollary 1.1.15) and theorem 1.1.40.

□

Definition 1.1.45. Let X be a good strictly k-analytic space. A subset $S \subset X$ is called **overconvergent subanalytic** if for all $x \in X$ there exists V a strictly affinoid neighbourhood of x such that $S \cap V$ is overconvergent subanalytic in V (according to definition 1.1.16).

According to the last proposition, when X is a k-affinoid space, this definition 1.1.45 is compatible with the previous one (definition 1.1.16).

Definition 1.1.46. Let X be a good strictly k-analytic space. A subset S of X is called **locally semianalytic** if for all $x \in X$ there exists V some strictly affinoid neighbourhood of x such that $V \cap S$ is semianalytic in V.

Corollary 1.1.47. *Let X be a good strictly k-analytic space. The class of locally semianalytic subsets of X is contained in the class of overconvergent constructible subsets of X.*

Corollary 1.1.48. *Let X be a strictly k-affinoid space and let $S \subset X$ be a subset of X. Then S is an overconvergent subanalytic subset of X if and only if there exist $r > 1$, an integer n, and $T \subseteq X \times \mathbb{B}^n_r$ a locally semianalytic subset, such that $S = \pi(T \cap (X \times \mathbb{B}^n))$.*

Proof. The first implication is true because a semianalytic subset of $X \times \mathbb{B}^n_r$ is in particular a locally semianalytic subset of $X \times \mathbb{B}^n_r$.

Conversely, if $S = \pi(T \cap (X \times \mathbb{B}^n))$ where T is a locally semianalytic subset of $X \times \mathbb{B}^n_r$, then according to corollary 1.1.47, T is overconvergent subanalytic in $X \times \mathbb{B}^n_r$, so according to corollary 1.1.42, $\pi(T \cap (X \times \mathbb{B}^n))$ is also overconvergent subanalytic. □

Lemma 1.1.49. *Let $\varphi : X \to Y$ be a morphism of good strictly k-analytic spaces, and $S \subseteq Y$ be a locally semianalytic subset. Then $\varphi^{-1}(S)$ is locally semianalytic.*

Proof. Let $x \in X$, $y = \varphi(x)$. There exists an affinoid neighbourhood V of Y such that $V \cap S$ is semianalytic in V. Let W be an affinoid neighbourhood of x in $\varphi^{-1}(V)$. Then $W \cap \varphi^{-1}(S)$ is semianalytic in W. □

Theorem 1.1.50. *Let $\varphi : Y \to X$ be a morphism of strictly k-affinoid spaces, and U an affinoid domain of Y such that $U \subseteq Int(Y/X)$. If S is an overconvergent subanalytic subset of Y then $\varphi(U \cap S)$ is an overconvergent subanalytic subset of X.*

Proof. According to [Ber90, Prop 2.5.9] there exist $\underline{r} > \underline{s} > 0$ and $\mathcal{A}\{\underline{r}^{-1}t\} \to \mathcal{B}$ an admissible epimorphism which hence identifies Y with a Zariski closed subset of $X \times \mathbb{B}_{\underline{r}}$, such that under this identification, $U \subseteq X \times \mathbb{B}_{\underline{s}}$. We can assume that $\underline{s} \in \sqrt{|k^\times|}^n$. If we denote by $\Gamma(\varphi)$ the graph of φ, this induces a Zariski closed embedding of $Y \simeq \Gamma(\varphi) \xrightarrow{i} X \times \mathbb{B}_{\underline{r}}$. Now since S is an overconvergent subanalytic subset of Y, according to lemma 1.1.18, $i(S)$ is an overconvergent subanalytic subset of $X \times \mathbb{B}_{\underline{r}}$). Finally, U is a semianalytic subset of Y (because of Gerritzen-Grauert theorem), so $i(U)$ is also semianalytic in $X \times \mathbb{B}_{\underline{r}}$, and by assumption, $i(U) \subseteq X \times \mathbb{B}_{\underline{s}}$, so $i(U \cap S) \subseteq X \times \mathbb{B}_{\underline{r}}$ is an overconvergent constructible subset of $X \times \mathbb{B}_{\underline{r}}$, and according to corollary 1.1.42, $\pi(i(U \cap S))$ is an overconvergent subanalytic subset of X. But this set is precisely $\varphi(U \cap S)$. □

Proposition 1.1.51. *Let $\varphi : Y \to X$ be a morphism of good strictly k-analytic spaces, S an overconvergent subanalytic subset of Y such that $\overline{|S|} \to |X|$ is proper and $\overline{|S|} \subseteq$ Int(Y/X). Then $\varphi(S)$ is an overconvergent subanalytic subset of X.*

Proof. If X' is an affinoid domain of X and if we consider the cartesian diagram :

$$\begin{array}{ccc} S \subseteq Y & \xrightarrow{\varphi} & X \\ {\scriptstyle \psi'} \uparrow & & \uparrow {\scriptstyle \psi} \\ S' \subseteq Y' & \xrightarrow{\varphi'} & X' \end{array}$$

then $\psi'^{-1}(\overline{S})$ is closed in Y' and contains $\psi'^{-1}(S) = S'$ so $S' \subseteq \overline{S'} \subseteq \psi'^{-1}(\overline{S})$, and since properness is stable under base change, $\psi'^{-1}(\overline{S}) \to |X'|$ is proper, and since $\overline{S'}$ is closed, $\overline{S'} \to |X'|$ is proper. Moreover, $\psi'^{-1}(\text{Int}(Y/X)) \subseteq \text{Int}(Y'/X')$ ([Ber90, 3.1.3 (iii)]) so $\overline{S'} \subseteq \psi'^{-1}(\overline{S}) \subseteq \text{Int}(Y'/X')$. So S' and φ' fulfil the hypotheses of the proposition. Hence, since the property we want to check is local on X, we can assume that X is a k-affinoid space, hence that $\overline{|S|}$ is compact.

Now for every $y \in \overline{|S|}$ we can find an affinoid neighbourhood U such that $U \subseteq \text{Int}(Y/X)$, because $\text{Int}(Y/X)$ is open. Then, $\varphi(U \cap S)$ is an overconvergent subanalytic subset of X according to theorem 1.1.50. Since $\overline{|S|}$ is compact we can extract from this a finite covering of $\overline{|S|}$, which finishes to prove that $\varphi(S)$ is overconvergent subanalytic. □

Corollary 1.1.52. *Let $\varphi : Y \to X$ be a proper morphism of good strictly k-analytic spaces. Remind ([Ber90, p.50]) that it means that $|\varphi| : |Y| \to |X|$ is proper and that $\partial(Y/X) = \emptyset$. Let S be an overconvergent subanalytic subset of Y. Then $\varphi(S)$ is an overconvergent subanalytic subset of X.*

Definition 1.1.53. A morphism $\varphi : Y \to X$ of good k-analytic spaces is locally extendible without boundary if, for all $y \in Y$, there exists an affinoid neighbourhood U of y, Y' a k-affinoid space that contains U as an affinoid domain, and $\psi : Y' \to X$ that extends $\varphi_{|U}$, such that $U \subseteq \text{Int}(Y'/X)$.

Remark that using again [Ber90, 3.1.3 (iii)], this property is stable under base change.

Proposition 1.1.54. *Let $\varphi : Y \to X$ be a compact morphism of good strictly k-analytic spaces (i.e. $|\varphi| : |Y| \to |X|$ is proper [Ber90, p.50]) which is locally extendible without boundary. Then $\varphi(Y)$ is an overconvergent subanalytic subset of X.*

Proof. We can assume that X is a k-affinoid space, so Y is compact. Then for all $y \in Y$ we can find an affinoid neighbourhood U of y and Y' a k-affinoid space that contains U, and $\psi : Y' \to X$ that extends $\varphi_{|U}$, such that $U \subseteq \text{Int}(Y'/X)$. Then, according to theorem 1.1.50, $\varphi(U)$ is an overconvergent subanalytic subset of X (take $S = Y'$). Hence by compactness of Y, $\varphi(Y)$ is overconvergent subanalytic. □

1.1.6 The non strict case

In this section, k will be an arbitrary non-Archimedean field (possibly trivially valued).

One of the advantages of Berkovich's approach is the possibility to use arbitrary $\lambda \in \mathbb{R}_+$ to define inequalities. It is then natural to give the following definitions:

Definition 1.1.55. Let $\mathcal{A}$ be a k-affinoid algebra, and let us set $X = \mathcal{M}(\mathcal{A})$.

1. A subset $S \subset X$ is called non-strictly semianalytic if it is a boolean combination of subsets
$$\{x \in X \mid \ |f(x)| \leq \lambda |g(x)|\}$$
where $f, g \in \mathcal{A}$ and $\lambda \in \mathbb{R}_+$.
2. A subset $S \subset X$ is called non-strictly overconvergent subanalytic if there exist an integer $n \in \mathbb{N}$, a real number $r > 1$, a non-strictly semianalytic set $T \subset X \times \mathbb{B}^n_r$ such that $S = \pi(T \cap (X \times \mathbb{B}^n))$ where $\pi : X \times \mathbb{B}^n_r \to X$ is the first projection.

Remark 1.1.56. Let X be a strictly k-affinoid space and let $S \subset X$. The following implication holds:
$$S \text{ is semianalytic} \Rightarrow S \text{ is non-strictly semianalytic.}$$
However, if $\sqrt{|k^\times|} \subsetneq \mathbb{R}^*_+$, the converse implication is false. Indeed, let $r \in]0, 1[$ such that $r \notin \sqrt{|k^\times|}$, let $X = \mathbb{B}^1 = \mathcal{M}(k\{t\})$ and let $S = \{x \in \mathbb{B} \mid |t(x)| = r\}$. By definition, S is a non-strictly semianalytic set of $\mathbb{B}^1$, but we claim that it is not semianalytic. Indeed, we will see in 1.2.13 that semianalytic sets are entirely determined by their rigid points, that is to say, if S_1 and S_2 are semianalytic subsets of X, then, $S_1 = S_2$ if and only if $S_1 \cap X_{rig} = S_2 \cap X_{rig}$. Since in our example, $S \cap X_{rig} = \emptyset$, if S was semianalytic, it would then be empty, but S is not empty. Actually $S = \{\eta_r\}$.

Definition 1.1.57. Let X be a k-affinoid space. Let (X, S) be a k-germ, $f, g \in \mathcal{A}$, $0 < s < r$ where $r, s \in \mathbb{R}$, and
$$Y = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f - tg)) \xrightarrow{\varphi} X$$
and $T = \varphi^{-1}(S) \cap R \cap \{y \in Y \mid |f(y)| \leq s|g(y)| \neq 0\}$ where R is a non-strictly semianalytic subset of Y. Then we say that $(Y, T) \xrightarrow{\varphi} (X, R)$ is a non-strictly elementary constructible datum.

The only difference with definition 1.1.3 is that we do not assume any more that $s \in \sqrt{|k^\times|}$, and that R is allowed to be non-strictly semianalytic, that is to say defined with inequalities involving some arbitrary $\lambda \in \mathbb{R}$.

Then we mimic definition 1.1.5, and say that a non-strictly constructible datum $(Y, T) \overset{\varphi}{\dashrightarrow} (X, S)$ is a composite $\varphi = \varphi_1 \circ \ldots \circ \varphi_n$ where each φ_i is a non-strictly elementary constructible datum. Finally, if $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X, i = 1 \ldots n$ are n non-strictly constructible data, we say that $S := \cup_{i=1}^n \varphi_i(S_i)$ is a non-strictly overconvergent constructible set.

We claim that all results we have proven in this section for overconvergent subanalytic (resp. constructible) sets remain valid for non-strictly overconvergent subanalytic (resp. constructible) sets. For instance:

Theorem 1.1.58. *Let X be a k-affinoid space. $S \subset X$ is non-strictly overconvergent subanalytic if and only if it is non-strictly overconvergent constructible.*

In this context, we want to stress out that for instance propositions 1.1.44, 1.1.51 also remain true.

1.1.7 A more *functional*-theoretical definition

In this section, we want to explain the links which exist between our definition of overconvergent constructible subsets and the notion of (strongly) D-semianalytic subsets which was studied in [Sch94a].

In this section $\mathcal{A}$ will be a k-affinoid algebra, and $X = \mathcal{M}(\mathcal{A})$ is the associated affinoid space.

Let k be a valued field. The operator D_k, that we will denote by D if no confusion is possible, is a function $k^2 \to k$ defined by:

$$D(x, y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \text{ and } |x| \leq |y| \\ 0 & \text{otherwise} \end{cases}$$

We will denote $k\{X_1 \dots X_n\}^\dagger$ the Monski-Washnitzer algebra [GK00] of overconvergent power series in n variables. By definition, it is the following inductive limit :

$$k\{X_1 \dots X_n\}^\dagger = \varinjlim_{r>1} k\{r^{-1}X_1 \dots r^{-1}X_n\}$$

Likewise we can define

$$\mathcal{A}\{T_1, \dots, T_n\}^\dagger := \varinjlim_{r>1} \mathcal{A}\{r^{-1}X_1 \dots r^{-1}X_n\}.$$

Schoutens has introduced [Sch94a, 1.2.2]) the following k-algebra:

Definition 1.1.59. $\mathcal{A}\langle\langle D\rangle\rangle$ is the smallest set of functions

$$f : \mathcal{M}(\mathcal{A}) \to \coprod_{x \in \mathcal{M}(\mathcal{A})} \mathcal{H}(x)$$

such that for all $x \in X$, $f(x) \in \mathcal{H}(x)$, which satisfies:

1. It contains the functions induced by $\mathcal{A}$.
2. $\mathcal{A}\langle\langle D\rangle\rangle$ is a k-subalgebra of the set of functions $f : \mathcal{M}(\mathcal{A}) \to \coprod_{x \in \mathcal{M}(\mathcal{A})} \mathcal{H}(x)$. In other words, if $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$, $f + g$ and $fg \in \mathcal{A}\langle\langle D\rangle\rangle$.
3. If f and $g \in \mathcal{A}\langle\langle D\rangle\rangle$, then $D(f, g)$ too where $D(f, g) = x \mapsto D_{\mathcal{H}(x)}(f(x), g(x))$.
4. If $F \in \mathcal{A}\{T_1 \dots T_n\}^\dagger$, and $f_1 \dots f_n \in \mathcal{A}\langle\langle D\rangle\rangle$, such that for all $i = 1 \dots n$, $\|f_i\| := \sup_{x \in \mathcal{M}(\mathcal{A})} |f_i(x)| \leq 1$, then $F(f_1 \dots f_n) \in \mathcal{A}\langle\langle D\rangle\rangle$.

The aim of this subsection is to prove proposition 1.1.67.

Remark 1.1.60. 1. It is easy to prove that in the fourth case of this definition, we could have allowed functions f_i such that $\sup_{x \in \mathcal{M}(\mathcal{A})} |f_i(x)| \leq r_i \in \sqrt{|k^*|}$ and $F \in \mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}^\dagger$. For simplicity we explain how it works for $n = 1$. In that case, we are given some $f \in \mathcal{A}\langle\langle D\rangle\rangle$ such that $\|f\| \leq r \in \sqrt{|k^*|}$ and $F \in \mathcal{A}\{r^{-1}T\}^\dagger$ and we want to prove that $F(f) \in \mathcal{A}\langle\langle D\rangle\rangle$. Let $N \in \mathbb{N}$ such that $r^N \in |k^*|$, and let $\lambda \in k^*$ such that $r^N = |\lambda|$. Moreover, let $\mu \in k^*$ such that $\|f\| \leq |\mu|$. We then introduce the following series:

$$G(X, Y) = \sum_{j \geq 0} \sum_{i=0}^{N-1} a_{Nj+i} \mu^i \lambda^j X^j Y^i.$$

It is easy to check that $G(X, Y) \in \mathcal{A}\{X, Y\}^\dagger$ and that $F(f) = G(\frac{f^N}{\lambda}, \frac{f}{\mu})$.

2. We also want to point out that if $\varphi : Y = \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A}) = X$ is a morphism of affinoid spaces, considering for all $x \in X$ the morphism $\mathcal{H}(\varphi(x)) \to \mathcal{H}(x)$, one can define a morphism $\varphi^* : \mathcal{A}\langle\langle D\rangle\rangle \to \mathcal{B}\{D\}^\dagger$, this construction being functorial.

3. Lastly, note that if $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$, then $\varphi^*(D(f,g)) = D(\varphi^*(f), \varphi^*(g))$.

Definition 1.1.61. Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space. A subset $S \subseteq X$ is called **overconvergent D-semianalytic** if it is a finite boolean combination of subsets of the form: $\{x \in \mathcal{M}(\mathcal{A}) \mid |f(x)| \bowtie |g(x)|\}$ where $\bowtie \in \{\leq, <, =\}$ and $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$.

In this definition one could have only authorized $\bowtie \in \{\leq\}$, and we will use this freely in the rest of this section.

Remark 1.1.62. Let (X,S) be a k-germ, $f, g \in \mathcal{A}$. Let us consider the following two elementary constructible data of (X,S):

Let $r > 1$, $s = 1$, and $\mathcal{B} = \mathcal{A}\{r^{-1}t\}/(f - tg)$, $Y = \mathcal{M}(\mathcal{B})$, φ the natural morphism $Y \xrightarrow{\varphi} X$, and $U = \{y \in Y \mid |f(y)| \leq |g(y)| \text{ and } g(y) \neq 0\} \cap \varphi^{-1}(S)$. Then $(Y,U) \xrightarrow{\varphi} (X,S)$ is an elementary constructible datum.

Secondly, let $V = \{x \in S \mid g(x) = 0 \text{ or } |f(x)| > |g(x)|\}$. Then $(X,V) \xrightarrow{id} (X,S)$ is also an elementary constructible datum. Moreover:

$S = V \coprod U$, and

– $id^*(D(f,g))_{|V} = 0 = 0_{|V}$

– $\varphi^*(D(f,g))_{|U} = t_{|U}$ where $t \in \mathcal{B}$.

In fact, $D(f,g)$ is not a function of the affinoid algebra of X, but should be thought of as a function that lives on some constructible data above X.

In all the statements and proofs which follow, if $(Y,T) \overset{\varphi}{\dashrightarrow} (X,S)$ is a constructible datum, we will constantly assimilate T and $\varphi(T)$ (see remark 1.1.4).

Proposition 1.1.63. *Let (X,S) be a germ, and $f \in \mathcal{A}\langle\langle D\rangle\rangle$. Then there exists an constructible covering*

$$(\mathcal{M}(\mathcal{B}_i), T_i) = (X_i, T_i) \overset{\varphi_i}{\dashrightarrow} (X,S),\ i = 1\ldots m$$

and for all i some functions $f_i \in \mathcal{B}_i$ such that $(\varphi_i^*(f))_{|T_i} = f_{i|T_i}$.

Proof. We show it by induction on the definition of f.

If $f \in \mathcal{A}$, this is clear, taking $\varphi = id$.

To prove the induction step we distinguish the cases which appear in definition 1.1.59.

Case $g + h$ and gh. For instance, if $g, h \in \mathcal{A}\langle\langle D\rangle\rangle$ and $f = gh$. If we assume that there exists a constructible covering $(X_i, T_i) \overset{\varphi_i}{\dashrightarrow} (X,S),\ i = 1\ldots m$ and some $g_i, h_i \in \mathcal{B}_i$ such that $(\varphi_i^*(g))_{|T_i} = g_{i|T_i}$ and $(\varphi_i^*(h))_{|T_i} = h_{i|T_i}$. Then we set $f_i := g_i h_i$ and then $(\varphi_i^*(f))_{|T_i} = f_{i|T_i}$ for all i.

Case $D(u,v)$. If $u, v \in \mathcal{A}\langle\langle D\rangle\rangle$ and we consider $D(u,v)$. Then by induction hypothesis and using corollary 1.1.12, one can find a constructible cover $(X_i, T_i) \overset{\varphi_i}{\dashrightarrow} (X,S), i = 1\ldots m$ and some functions $u_i, v_i \in \mathcal{B}_i$, such that $\varphi_i^*(u)_{|T_i} = u_{i|T_i}$, and $\varphi_i^*(v)_{|T_i} = v_{i|T_i}$ for all i.

We can then conclude with remark 1.1.62.

Case $F(f_1, \ldots f_n)$. If $f_i \in \mathcal{A}\langle\langle D\rangle\rangle$, $i = 1\ldots n$ are n functions such that $\|f_i\| \leq 1$, and $F \in \mathcal{A}\{X_1, \ldots X_n\}^\dagger$, we want to treat the case of $F(f_1 \ldots f_n) \in \mathcal{A}\langle\langle D\rangle\rangle$. Let $r > 1$ such that $F \in \mathcal{A}\{r^{-1}T_1, \ldots, r^{-1}T_n\}$. By induction hypothesis we have in hand a constructible covering of (X,S), $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} (X,S)$, $i = 1\ldots m$, and some functions $f_{1,i} \ldots f_{n,i} \in \mathcal{A}_i = \mathcal{O}(X_i)$ such that $f_{k,i|S_i} = \varphi_i^*(f_k)_{|S_i}$. For $i = 1\ldots m$, $k = 1\ldots n$, we set $\mathcal{A}_i^k = \mathcal{A}_i\{r^{-1}T_{k,i}\}/(f_{k,i} - T_{k,i})$, $X_i^k = \mathcal{M}(\mathcal{A}_i^k)$, and $S_i^k = \{x \in X_i^k \mid |f_{k,i}(x)| \leq 1\}$. $(X_i^k, S_i^k) \overset{\varphi_i^k}{\dashrightarrow} (X,S)$ is then a constructible cover. We then define for each $i = 1\ldots m$ a constructible datum: $(Y_i, T_i) \overset{\psi_i}{\dashrightarrow} (X,S)$ which is the fibered product of all the constructible data

$(X_i^k, S_i^k) \overset{\varphi_i^k}{\dashrightarrow} (X,S)$, $k = 1 \dots n$. Then one checks that on Y_i the function $F(T_{i,1}, \dots, T_{i,n})$ is well defined and has the expected properties. □

Lemma 1.1.64. *Let $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$. If we consider the functions $\mathbb{1}_B$, where B is one of the D-semianalytic subset $\{x \in X \mid |f(x)| = 0\}$, $\{x \in X \mid |g(x)| \leq |f(x)|\}$ and $\{x \in X \mid |g(x)| < |f(x)|\}$. Then $\mathbb{1}_B \in \mathcal{A}\langle\langle D\rangle\rangle$.*

Proof. First let us note that $D(f,f) = \mathbb{1}_{f\neq 0}$. So $1 - D(f,f) = \mathbb{1}_{f=0} \in \mathcal{A}\langle\langle D\rangle\rangle$. In particular $D(D(f,g), D(f,g)) = \mathbb{1}_{D(f,g)\neq 0}$. Now $D(f,g)(x) \neq 0$ is equivalent to $g(x) \neq 0$ and $|f(x)| \leq |g(x)|$ and $f(x) \neq 0$, which is also equivalent to $f(x) \neq 0$ and $|f(x)| \leq |g(x)|$. So $D(D(f,g), D(f,g)) = \mathbb{1}_{\{f\neq 0 \text{ and } |f|\leq|g|\}}$.

Likewise, one checks that $D(f,g)D(g,f) = \mathbb{1}_{|g|=|f|>0}$. Finally one can partition X in the following way:

$$\begin{array}{llll} X = & X_1 \coprod & X_2 \coprod & X_3 \coprod X_4 \\ = \{f = 0\} & \coprod\{f \neq 0 \text{ and } |g| = |f|\} & \coprod\{f \neq 0 \text{ and } |g| > |f|\} & \coprod\{f \neq 0 \text{ and } |g| < |f|\} \end{array}$$

which gives $1 = \mathbb{1}_{X_1} + \mathbb{1}_{X_2} + \mathbb{1}_{X_3} + \mathbb{1}_{X_4}$. We have seen that $\mathbb{1}_{X_1}$ and $\mathbb{1}_{X_2} \in \mathcal{A}\langle\langle D\rangle\rangle$. Moreover $\mathbb{1}_{X_2} + \mathbb{1}_{X_3} = \mathbb{1}_{\{f\neq 0 \text{ and } |g|\geq|f|\}} \in \mathcal{A}\langle\langle D\rangle\rangle$. Hence $\mathbb{1}_{X_3} \in \mathcal{A}\langle\langle D\rangle\rangle$ and $\mathbb{1}_{X_4}$ too. □

Lemma 1.1.65. *If $s \in \sqrt{|k^*|}$, and $f, g \in \mathcal{A}\langle\langle D\rangle\rangle$, let us define*

$$\begin{array}{rcll} D_s(f,g) & : & X & \to \coprod_{x\in X} \mathcal{H}(x) \\ & & x & \mapsto \begin{cases} \frac{f(x)}{g(x)} & \text{if } g(x) \neq 0 \text{ and } |f(x)| \leq s|g(x)| \\ 0 & \text{otherwise} \end{cases} \end{array}$$

Then $D_s(f,g) \in \mathcal{A}\langle\langle D\rangle\rangle$ and $\|D_s(f,g)\| \leq s$.

Proof. Let $\mu \in k^*$ such that $|\mu| \geq s$, and $\lambda \in k^*$, $d \in \mathbb{N}^*$ such that $s^d = |\lambda|$. For $x \in X$,

$$(\mu D(f, g\mu))(x) = \begin{cases} \frac{f(x)}{g(x)} & \text{if } g(x) \neq 0 \text{ and } |f(x)| \leq |\mu||g(x)| \\ 0 & \text{otherwise} \end{cases}$$

Let $U = \{x \in X \mid |f(x)| \leq s|g(x)|\} = \{x \in X \mid |f^d(x)| \leq |(\lambda g^d)(x)|\}$. According to lemma 1.1.64, $\mathbb{1}_U \in \mathcal{A}\langle\langle D\rangle\rangle$, so $D_s(f,g) = \mathbb{1}_U \mu D(f, \mu g)$. □

Lemma 1.1.66. *Let $(Y,T) \xrightarrow{\varphi} (X,S)$ be an elementary constructible datum with $Y = \mathcal{M}(\mathcal{B})$. Let $h \in \mathcal{B}\{D\}^\dagger$. There exists $h' \in \mathcal{A}\langle\langle D\rangle\rangle$ such that $h_{|T} = \varphi^*(h')_{|T}$.*

Proof. There exist $f, g \in \mathcal{A}$, some real numbers $r > s > 0$, $d \in \mathbb{N}$, $\lambda \in k^*$, such that $s^d = |\lambda|$, and $\mathcal{B} = \mathcal{A}\{r^{-1}t\}/(f - tg)$.

We then make an induction on the number of steps in the definition of h.

If there is 0 step, $h \in \mathcal{B}$, so h has a representative $F(t) = \sum_{n\geq 0} a_n t^n$ in $\mathcal{A}\{r^{-1}t\}$. We then define $h' = F(D_s(f,g))$. Then using lemma 1.1.65 $\varphi^*(h')_{|T} = h_{|T}$ suits.

To make the induction step, if $h = h_1 + h_2, h_1.h_2$, $\mu.h_1$ or $D(h_1, h_2)$ this works easily.

If $h = F(f_1, \dots, f_n)$, with $f_i \in \mathcal{B}\{D\}^\dagger$ such that $\|f_i\| \leq 1$, and $F \in \mathcal{B}\{X_1, \dots, X_n\}^\dagger$, i.e. $F = \sum_{\nu\in\mathbb{N}^n} b_\nu X^\nu$ such that $\|b_\nu \underline{r}^\nu\| \xrightarrow[|\nu|\to\infty]{} 0$ for a polyradius $\underline{r} > 1$. Since ideals in affinoid algebras are strictly closed [BGR84, corollary 5.2.7.8], and $\mathcal{B} = \mathcal{A}\{r^{-1}t\}/(f - tg)$, for all $\nu \in \mathbb{N}^n$, there exists a series $\sum_{m\geq 0} a_{\nu,m}t^m$, that is a representative of b_ν in $\mathcal{A}\{r^{-1}t\}$, such that $\|b_\nu\| = \|\sum_{m\geq 0} a_{\nu,m}X^m\|_{\mathcal{A}\{r^{-1}t\}} = \max_{m\geq 0}\|a_{\nu,m}\|r^m$. Then we define

$$G(X_1,\dots,X_n,t) = \sum_{\nu=(\nu_1,\dots,\nu_n,m)\in\mathbb{N}^{n+1}} a_{(\nu_1,\dots,\nu_n),m}X^\nu t^m$$

Then $G \in \mathcal{A}\{r_1^{-1}X_1,\dots,r_n^{-1}X_n,r^{-1}t\}$, and since $(r_1,\dots,r_n,r) > (1,\dots,1,s)$, $G \in \mathcal{A}\{X_1,\dots,X_n,s^{-1}X\}^\dagger$. Hence using remark 1.1.60, $h' = G(f_1,\dots,f_n,D_s(f,g)) \in \mathcal{A}\langle\langle D\rangle\rangle$. One checks that $h_{|T} = \varphi^*(h')_{|T}$. □

Proposition 1.1.67. *Let $X = \mathcal{M}(\mathcal{A})$. $S \subseteq X$ is overconvergent D-semianalytic if and only if it is an overconvergent constructible subset of X.*

Proof. Let $C \subset X$ be an overconvergent D-semianalytic subset, and let us show that it is an overconvergent constructible subset. Since overconvergent constructible subsets are stable under finite boolean combination, we can assume that $C = \{x \in X \mid |f(x)| \leq |g(x)|\}$ where $f,g \in \mathcal{A}\langle\langle D\rangle\rangle$. According to proposition 1.1.63, we can find a constructible covering of X, $(X_i,S_i) \overset{\varphi_i}{\dashrightarrow} X$, $i = 1\dots m$, with $X_i = \mathcal{M}(\mathcal{B}_i)$, and some functions $f_i, g_i \in \mathcal{B}_i$ such that $f_{i|S_i} = \varphi_i^*(f)_{|S_i}$ and $g_{i|S_i} = \varphi_i^*(g)_{|S_i}$ for all $i = 1\dots m$. For each i we define the constructible datum $(X_i,T_i) \overset{\varphi_i}{\dashrightarrow} X$ where $T_i = \{x \in S_i \mid |f_i(x)| \leq |g_i(x)|\}$, and since we initially had a constructible cover, $C = \bigcup_{i=1}^n \psi_i(T_i)$ so C is an overconvergent constructible subset of X.

Conversely, if $C \subset X$ is an overconvergent constructible subset of X, let us show that it is overconvergent D-semianalytic. Let us actually prove that if $(Y,T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum and if $U \subseteq Y$ is an overconvergent D-semianalytic subset of Y, then $\varphi(U \cap T)$ is overconvergent D-semianalytic in X (this is more than enough since the class of D-semianalytic subset contains the class of semianalytic subsets). In order to prove this, we make an induction on the length of the constructible datum, and can then restrict to an elementary constructible datum $(Y,T) \overset{\varphi}{\to} X$, where $Y = \mathcal{M}(\mathcal{B})$, and suppose that $U \subseteq Y$ is overconvergent D-semianalytic. For simplicity, let us assume that $U = \{y \in Y \mid |f(y)| \leq |g(y)|\}$ with $f,g \in \mathcal{B}\{D\}^\dagger$. Then according to lemma 1.1.66 we can find functions $f',g' \in \mathcal{A}\langle\langle D\rangle\rangle$ that lift $f_{|T}$ and $g_{|T}$. Then $\varphi(T\cap U) = \varphi(T)\cap C$ where $C = \{x \in X \mid |f'(x)| \leq |g'(x)|\}$ hence is overconvergent D-semianalytic. □

1.2 Study of various classes

1.2.1 Many families

In this section $X = \mathcal{M}(\mathcal{A})$ will be a strictly k-affinoid space. The aim of this section is to first recall the definitions of the various classes of *rigid/locally/strongly/D-semianalytic/subanalytic* subsets of X that are defined in [Sch94a].

We now give the following definitions. A subset $S \subseteq X$ is called :

(a) semianalytic if it is a boolean combination of subsets of the form $\{x \in X \mid |f(x)| \leq |g(x)|\}$, with $f,g \in \mathcal{A}$.

(b) Locally semianalytic, if for all $x \in X$ there exists an affinoid neighbourhood V of x such that $S \cap V$ is semianalytic in V.

(c) rigid semianalytic if there is a finite affinoid covering $\{X_i\}_{i=1}^n$ such that $S \cap X_i$ is semianalytic in X_i for all i.

(d) Overconvergent subanalytic has been defined in definition 1.1.8. As we proved in the previous section, this corresponds also to overconvergent constructible subsets. Moreover, our definition of overconvergent subanalytic subset is the same as the definition of globally strongly subanalytic of [Sch94a, 1.3.8.1]. In [Sch94a] it is proven and it is correct that this is equivalent to the class of globally strongly **D**-semianalytic subsets [Sch94a, 1.3.2].

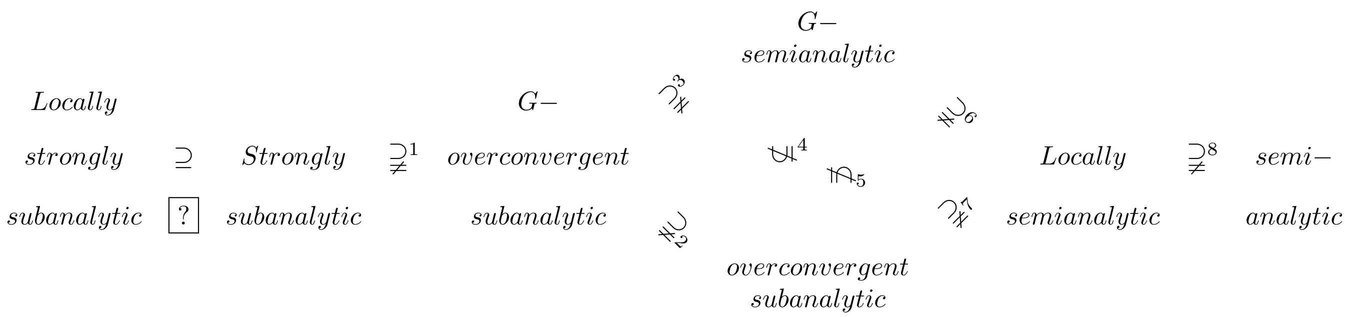
(e) G-overconvergent subanalytic if there exists a finite affinoid covering $\{X_i\}$ of X such that $S \cap X_i$ is overconvergent constructible in X_i for all i. This corresponds to the notion of *strongly* **D** *-semianalytic* subset in [Sch94a, 1.3.7.1].

(f) Strongly subanalytic if there exist an integer n, $r > 1$, a subset $T \subseteq X \times \mathbb{B}_r^n$ which is rigid semianalytic, such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. This is definition [Sch94a, 1.3.8.1], and we will give an equivalent definition in proposition 1.2.8.

(g) Locally strongly subanalytic if there exists a finite affinoid covering $\{X_i\}$ of X such that $S \cap X_i$ is strongly subanalytic in X_i for all i. This is definition [Sch94a, 1.3.8.2].

In [Sch94a] it is stated that (d),(e),(f) and (g) are equivalent (equivalence of $(e), (f), (g)$ is stated in [Sch94a, Prop 4.2], and the equivalence of (d) and (f) is stated in [Sch94a, Th 5.2]). These results rest on [Sch94a, lemma 4.1] which is false, and we will show indeed that $(d), (e)$ and (f) correspond in general to three different classes. More precisely the aim of this section is to show that these classes satisfy the following relations:

Figure 1.1: The hierarchy



In this figure, Class A $\supsetneq$ Class B, means that the class A *properly* contains the class B, and Class A $\not\supseteq$ Class B means that the Class A does not contain the class B.

In this diagram, all the inclusions are clear from the definitions, except the inclusion 7 which states that the class of overconvergent subanalytic subsets contains the class of locally semianalytic subsets. But this is precisely the content of corollary 1.1.44. In comparison with what was stated in [Sch94a], the most striking inequality is probably $\not\subseteq^5$ which asserts that rigid semianalytic subsets are not necessarily overconvergent subanalytic subsets whereas according to [Sch94a, Th 5.2], they should be overconvergent subanalytic. In other words, when you project overconvergent semianalytic subsets, you obtain a class which is not G-local (but however local for the Berkovich topology).

In this section we will show that the inclusions (1)-(8) in figure 1.1 are all proper in general (in the next section we will explain that if X is regular of dimension 2, overconvergent subanalytic subsets correspond to locally semianalytic subsets). We do not know if the inclusion on the left

$$\textit{locally strongly subanalytic} \supseteq \textit{strongly subanalytic}$$

is proper.

1.2.2 rigid semianalytic sets are not necessarily overconvergent subanalytic

Here we prove the inequality $\not\subseteq^5$.

Lemma 1.2.1. *Let $\eta \in X$ such that $\mathcal{O}_{X,\eta}$ is a field, $S \subset X$ a semianalytic subset. If $\eta \in \overline{S}$, then $\overset{\circ}{S}$ is non empty.*

Proof. Since

$$\bigcup_{i=1}^{n} \overline{S_i} = \overline{\bigcup_{i=1}^{n} S_i}$$

we can assume that S is a basic semianalytic subset, i.e is of the form:

$$S = \left(\bigcap_{i=1}^{m} \{x \in X \mid |f_i(x)| \leq |g_i(x)|\} \right) \bigcap \left(\bigcap_{j=1}^{n} \{x \in X \mid |F_j(x)| < |G_j(x)|\} \right).$$

We use the following decomposition

$$\{x \in X \mid |f_i(x)| \leq |g_i(x)|\} = \{x \in X \mid f_i(x) = g_i(x) = 0\} \cup \{x \in X \mid \ |f_i(x)| \leq |g_i(x)| \neq 0\}$$

and using again that the adherence is stable under finite union, we can assume that $\eta \in \overline{S}$ and that S is of the form:

$$S = \bigcap_{i=1}^{l} \{x \in X \mid h_i(x) = 0\} \bigcap_{j=1}^{m} \{x \in X \mid \ |f_j(x)| \leq |g_j(x)| \neq 0\} \bigcap_{k=1}^{n} \{x \in X \mid |F_k(x)| < |G_k(x)|\}.$$

Since the subsets $\{x \in X \mid h_i(x) = 0\}$ are closed, contain S and $\eta \in \overline{S}$, it follows that $h_i(\eta) = 0$.

Since $\mathcal{O}_{X,\eta}$ is a field we can find an affinoid neighbourhood V of η such that $h_{i|V} = 0$ for all i. Hence $V \cap S \neq \emptyset$ (because $\eta \in \overline{S}$) and we can remove the $h_i's$, and assume that

$$V \cap S = \bigcap_{j=1}^{m} \{x \in V \mid |f_j(x)| \leq |g_j(x)| \neq 0\} \bigcap_{k=1}^{n} \{x \in V \mid |F_k(x)| < |G_k(x)|\}.$$

This defines a strictly k-analytic domain of X, which is non empty, so its interior is also non empty, for instance its interior contains some rigid points. □

Lemma 1.2.2. *Let $\eta \in X$ and let us assume that $\mathcal{O}_{X,\eta}$ is a field. Let $(Y,T) \xrightarrow{\varphi} (X,S)$ be an elementary constructible datum with $Y = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f-tg))$ where $T = \varphi^{-1}(S) \cap \{y \in R \mid |f(y)| \leq s|g(y)| \neq 0\}$ with $0 < s < r$, $s \in \sqrt{|k^\times|}$ and R a semianalytic subset of Y. Let us assume that $\eta \in \overline{\varphi(T)}$. Then*

(a) $g(\eta) \neq 0$.

(b) $|f(\eta)| \leq s|g(\eta)|$.

(c) There exists a neighbourhood U of η such that $\varphi^{-1}(U) \xrightarrow{\varphi_{|\varphi^{-1}(U)}} U$ is an isomorphism. If we denote by η' the only point of $\varphi^{-1}(U)$ such that $\varphi(\eta') = \eta$, then $\eta' \in \overline{T}$ and $\mathcal{O}_{Y,\eta'}$ is a field.

Proof.

(a) If we had $g(\eta) = 0$, since $\mathcal{O}_{X,\eta}$ is a field, there would exist an affinoid neighbourhood of η, V, such that $g_{|V} = 0$. Since for $p \in T$, $g(\varphi(p)) \neq 0$ we should have $\varphi(T) \cap V = \emptyset$ which is impossible since $\eta \in \overline{\varphi(T)}$.

(b) The subset $\{x \in X \mid |f(x)| \leq s|g(x)|\}$ is a closed subset of X which contains $\varphi(T)$, hence by assumption also η.

(c) If we set $U = \{y \in Y \mid g(y) \neq 0\}$, $\varphi_{|U}$ identifies through an isomorphism U with $\varphi(U) = \{x \in X \mid |f(x)| \leq r|g(x)| \neq 0\}$ which is an analytic domain of X, and a neighbourhood of η according to the two preceding points. So $\eta \in \varphi(U)$, let us say $\eta = \varphi(\eta')$ with $\eta' \in U$. Now, $\mathcal{O}_{Y,\eta'} \simeq \mathcal{O}_{X,\eta}$ is a field and $\eta' \in \overline{T}$.

□

Corollary 1.2.3. *Let $\eta \in X$ such that $\mathcal{O}_{X,\eta}$ is a field, and let U be an overconvergent subanalytic subset of X. If $\eta \in \overline{U}$, then $\mathring{U} \neq \emptyset$.*

Proof. First, according to theorem 1.1.38, we can assume that U is an overconvergent constructible subset. Then, using similar arguments as in the beginning of lemma 1.2.1, we can assume that $U = \varphi(T)$ where $(Y,T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum. Hence T is a semianalytic subset of Y. A repeated use of lemma 1.2.2 allows us to say that there exists an open neighbourhood U of η, such that $\varphi^{-1}(U) \xrightarrow{\varphi_{|\varphi^{-1}(U)}} U$ is an isomorphism. Thanks to lemma 1.2.2 again, we can introduce η', the only point of $\varphi^{-1}(U)$ such that $\varphi(\eta') = \eta$, and assert that $\mathcal{O}_{Y,\eta'}$ is a field and that $\eta' \in \overline{T}$. Now if we consider V a strictly affinoid neighbourhood of η' contained in $\varphi^{-1}(U)$, it is true that $\eta' \in \overline{T \cap V}$ (the adherence is here taken in V). Now, $T \cap V$ is a semianalytic subset of V so according to lemma 1.2.1, $T \cap V$ has non empty interior in V. We can then deduce that T has non empty interior in X whence $\varphi(T)$ has also non-empty interior. □

Proposition 1.2.4. *Let $X = \mathbb{B}^2 = \mathcal{M}(k\{T_1, T_2\})$ be the closed bidisc, and let $0 < r < 1$ with $r \in |k^*|$, say $r = |\varepsilon|$ for some $\varepsilon \in k$, and let $f \in k\{r^{-1}u\}$ be some function whose radius of convergence is exactly r, and $\|f\| < 1$. We then define*

$$S = \{x \in X \mid |T_1(x)| < r \textit{ and } T_2(x) = f(T_1(x))\}.$$

Then S is rigid semianalytic but not overconvergent subanalytic. As a consequence, the class of overconvergent subanalytic subsets is not G-local.

Proof. In more concrete terms, S is the set of points of the *curve* whose equation is $T_2 = f(T_1)$, restricted to the subset $\{|T_1| < r\}$. Let us consider

$$\begin{array}{ccc} \mathbb{B} & \overset{\psi}{\to} & X \\ u & \mapsto & (\varepsilon u, f(\varepsilon u)) \end{array}$$

and let us set $\eta = \psi(g)$ where g is the Gauss point of $\mathbb{B}$. Then $S \subseteq \psi(\mathbb{B})$ and $\eta \in \overline{S}$. According to [Duc, lemma 2.21] $\mathcal{O}_{X,\eta}$ is a field. Furthermore $\mathring{S} = \emptyset$ because $S \subseteq Z := \{x \in \mathbb{B}_{(r,1)} \mid T_2(x) = f(T_1(x))\}$, which is a Zariski closed subset of dimension 1 of $\mathbb{B}_{(r,1)}$, which itself is of pure dimension 2, so Z is nowhere dense in $\mathbb{B}_{(r,1)}$ [Ber90, 2.3.7]. Hence according to corollary 1.2.3, S is not overconvergent subanalytic. However, if we consider the covering of X given by $X_1 = \{x \in X \mid |T_1(x)| \leq r\}$, $X_2 = \{x \in X \mid |T_1(x)| \geq r\}$, then $S \cap X_1$ is indeed semianalytic in X_1 and $S \cap X_2 = \emptyset$, so S is rigid semianalytic.

Now since the class of overconvergent subanalytic subsets contains the class of semianalytic subsets, if the class of overconvergent subanalytic subsets was G-local, it should contain the class of rigid semianalytic subsets, but we have shown that this is false. Hence the class of overconvergent subanalytic subsets is not G-local. □

Remark 1.2.5. Actually, this example gives directly a counterexample to [Sch94a, lemma 4.1] which in our feeling is the source of mistakes in [Sch94a].

As a corollary of this we obtain:

Proposition 1.2.6. *Let $0 < s < r < 1$ with $s \in \sqrt{|k^\times|}$, $f \in k\{r^{-1}u\}$ whose radius of convergence is exactly r such that $\|f\| < 1$, and let us set $\mathbb{B}^2 = \mathcal{M}(k\{T_1, T_2\})$. Define :*

$$S = \{x \in \mathbb{B}^2 \mid |T_1(x)| \leq s \text{ and } T_2(x) = f(T_1(x))\}.$$

Then S is a locally semianalytic subset of $\mathbb{B}^2$ which is not a semianalytic subset of $\mathbb{B}^2$.

Proof. If S was a semianalytic subset of $\mathbb{B}^2$, we could find $T \subseteq S$ which contains infinitely many points of S such that T is a basic semianalytic subset, and even, a finite intersection of sets of the form $\{x \in \mathbb{B}^2 \mid |g_1(x)| < |g_2(x)|\}$, $\{x \in \mathbb{B}^2 \mid |g_1(x)| \leq |g_2(x)| \neq 0\}$ and $\{x \in \mathbb{B}^2 \mid h(x) = 0\}$. Since an intersection of the two first kind of sets is a strictly analytic domain, and $T \subseteq S$, and $\overset{\circ}{S} = \emptyset$, in this intersection, there must be a non-trivial set of the form $\{x \in \mathbb{B}^2 \mid h(x) = 0\}$. Now, let us consider in $\mathbb{B}_{(r,1)} = \mathcal{M}(k\{r^{-1}T_1, T_2\})$ the Zariski-closed subset $Z = V(T_2 - f(T_1), h)$. By assumption, it is infinite. Moreover, since $\|f\| < 1$, $T_2 - f(T_1)$ is irreducible (see the lemma above) in $\mathcal{M}(k\{r^{-1}T_1, T_2\})$, so for reasons of dimension, in $\mathcal{M}(k\{r^{-1}T_1, T_2\})$, $V(T_2 - f(T_1)) \subseteq V(h)$. But now if we introduce (as in the preceding proof)

$$\begin{array}{rcl} \mathbb{B}_r & \xrightarrow{\psi} & \mathbb{B}^2 \\ u & \mapsto & (u, f(u)) \end{array}$$

and $\eta = \psi(g)$ where g is the Gauss point of $\mathbb{B}_r$, then $\eta \in V(h)$ (where we see now $V(h)$ as a Zariski closed subset of $\mathbb{B}^2$), $\mathcal{O}_{\mathbb{B}^2,\eta}$ is a field, but $\overset{\circ}{V(h)} = \emptyset$, and since $V(h)$ is a semianalytic (so overconvergent subanalytic) subset of $\mathbb{B}_{(r,1)}$, this contradicts lemma 1.2.1.

Let us now show that S is a locally semianalytic subset of $\mathbb{B}^2$. Indeed, take $0 < s < t < r$ with $t, r \in \sqrt{|k^\times|}$, and consider $X_1 = \{x \in \mathbb{B}^2 \mid |T_1(x)| \leq r\}$ and $X_2 = \{x \in \mathbb{B}^2 \mid |T_1(x)| \geq t\}$. They define a wide covering of $\mathbb{B}^2$ and $X_1 \cap S$ (resp. $X_2 \cap S$) is semianalytic in X_1 (resp. X_2), so S is well locally semianalytic in $\mathbb{B}^2$. □

We have implicitly used:

Lemma 1.2.7. *If $f \in k\{r^{-1}x\}$ and $\|f\| \leq 1$, then $F(x, y) := y - f(x)$ is irreducible in $k\{r^{-1}x, y\}$.*

Proof. As we have already seen, $V(f)$ is isomorphic to $\mathbb{B}_r$, so is irreducible. □

1.2.3 The other inequalities

We will now explain the other inequalities appearing in figure 1.

The following proposition will be implicitly used in the rest of this section. In addition, it illustrates that the mixture of overconvergence and rigid semianalytic subsets (which is a G-local property), is somehow too strong, in the sense that in proposition 1.2.8 above, the overconvergence condition seems to have disappeared.

Proposition 1.2.8. *Let $S \subseteq X$. The following properties are equivalent:*

1. *S is strongly subanalytic.*
2. *There exist $n \in \mathbb{N}$ and $T \subseteq X \times \mathbb{B}^n$ a rigid semianalytic subset such that*

$$S = \pi(T \cap (X \times (\overset{\circ}{\mathbb{B}})^n))$$

where $\pi : X \times \mathbb{B}^n \to X$ is the natural projection.

Proof. Let us show that (1) $\Rightarrow$ (2). Let S be a strongly subanalytic subset of X, so there exists $r > 1$, $T \subseteq X \times \mathbb{B}^n_r$ a rigid semianalytic subset such that $S = \pi(T \cap (X \times \mathbb{B}^n))$. Decreasing r if necessary, we can assume that $|r| \in \sqrt{|k^*|}$. In fact, using similar arguments as the one given in remark 1.1.19, we can even assume that $r \in |k|$. Then if we consider the homothety, which is an isomorphism: $h : X \times \mathbb{B}^n_r \to X \times \mathbb{B}^n$, which can be defined as multiplication of each coordinate of $\mathbb{B}^n_r$ by $\frac{1}{\lambda}$, this gives the following commutative diagram:

$$\begin{array}{ccc} X \times \mathbb{B}^n_r & \xrightarrow{h} & X \times \mathbb{B}^n \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi'} \\ & & X \end{array}$$

and $S = \pi(T \cap (X \times \mathbb{B}^n) = \pi'\left(h(T) \cap (X \times \mathbb{B}^n_{\frac{1}{r}})\right)$. Now $T' := h(T) \cap (X \times \mathbb{B}^n_{\frac{1}{r}})$ is a rigid semianalytic subset of $X \times \mathbb{B}^n$ such that $T' \subseteq X \times (\overset{\circ}{\mathbb{B}})^n$ and $S = \pi'(T') = \pi'(T' \cap (X \times (\overset{\circ}{\mathbb{B}})^n))$.

Conversely, let $T \subseteq X \times (\overset{\circ}{\mathbb{B}})^n$ be a rigid semianalytic subset of $X \times \mathbb{B}^n$ and $S = \pi(T)$. For any $r > 1$, we can define $X_0 = X \times \mathbb{B}$, and for $i = 1 \dots n$, let $X_i = \{(x, t_1, \dots, t_n) \in X \times \mathbb{B}^n_r \mid |t_i| \geq 1\}$. So $\{X_i\}_{i \in \{0 \dots n\}}$ is an admissible covering of $X \times \mathbb{B}^n_r$. By assumption, $T \cap X_0$ is rigid semianalytic, and $T \cap X_i = \emptyset$ for $i = 1 \dots n$. So T is rigid semianalytic in $X \times \mathbb{B}^n_r$, and if we note $\pi : X \times \mathbb{B}^n_r \to X$, $S = \pi(T)$, so S is strongly subanalytic. □

Proposition 1.2.9. *There exist strongly subanalytic subsets which are not G-overconvergent subanalytic.*

Proof. Let $r > 1$, $X = \mathcal{M}(k\{x, y, z\}) = \mathbb{B}^3$, $Y = \mathcal{M}(k\{x, y, z, t\})$ and $\pi : Y = \mathcal{M}(k\{x, y, z, t\}) \to X = \mathcal{M}(k\{x, y, z\})$, the natural projection. We now choose $f \in k\{t\}$ whose radius of convergence is exactly 1, and such that $\|f\| \leq 1$, and $T = \{(x, y, z, t) \in Y \mid |t| < 1, x = yt, z = yf(t)\}$. It is a rigid semianalytic subset of Y, and $S = \pi(T)$ is a strongly subanalytic subset of X according to the previous proposition. Since the family of closed balls with center the origin is a fundamental system of neighbourhoods of the origin, if S was G-overconvergent subanalytic, for some $1 \geq |\mu| = \varepsilon > 0$ small enough, $S' := S \cap \mathbb{B}^3_\varepsilon$ would be overconvergent subanalytic in $\mathbb{B}^3_\varepsilon$. We then fix a $y_0 \in k^*$ such that $0 < |y_0| < \varepsilon$, i.e. $\frac{|y_0|}{|\mu|} < 1$ and define $X' := \{(x, y, z) \in \mathbb{B}^3_\varepsilon \mid y = y_0\}$. Now X' is isomorphic to the bidisc $\mathbb{B}^2_\varepsilon = \{(x, y) \mid |x| \leq \varepsilon$ and $|y| \leq \varepsilon\}$, and $S'' := S \cap X'$ should be overconvergent constructible in X' (somehow we use here lemma 1.1.18 (2)). If we make a dilatation of X' by $\frac{1}{\mu}$ it becomes the bidisc of radius 1: the new coordinates are x', z' defined by $x = \mu x'$ and $z = \mu z'$. Now, in these new coordinates:

$$S'' = \{(x', z') \in \mathbb{B}^2 \mid |x'| < \frac{|y_0|}{|\mu|} \text{ and } z' = \frac{y_0}{\mu} f(\frac{x'\mu}{y_0})\}$$

should be overconvergent subanalytic in $\mathbb{B}^2$. If we put $r := \frac{|y_0|}{|\mu|} < 1$ and $g(x') = \frac{y_0}{\mu} f(\frac{x'\mu}{y_0})$, then the radius of convergence of g is precisely r, $\|g\| < 1$ so $S'' = \{(x', z') \in \mathbb{B}^2 \mid |x'| <$

r and $z' = g(x')\}$, S'' should be overconvergent subanalytic in $\mathbb{B}^2$, but we proved the converse in proposition 1.2.4. □

Proposition 1.2.10. *There exist overconvergent subanalytic subsets which are not rigid semianalytic.*

Proof. Let $1 < r = |\lambda|$, and $f \in k\{r^{-1}X\}$ whose radius of convergence is exactly r, and such that $\|f\| < 1$. We set $X = \mathbb{B}^3 = \mathcal{M}(k\{x,y,z\})$, $Y = \mathcal{M}(k\{x,y,z,r^{-1}t\})$, and

$$T = \{(x,y,z,t) \in Y \mid x = yt,\ z = yf(t),\ |t| \leq 1\}$$

and $S = \pi(T)$, where $\pi : \mathcal{M}(k\{x,y,z,r^{-1}t\}) \to \mathcal{M}(k\{x,y,z\})$ is the natural projection. Then S is overconvergent subanalytic. If S was rigid semianalytic, there would exist $\mu \in k$, with $0 < \varepsilon := |\mu| < 1$ such that $S' = S \cap \mathbb{B}^3$ is semianalytic in $\mathbb{B}^3_\varepsilon$ (we *again* use that if V is an affinoid domain of $\mathbb{B}^3$ that contains the origin, then there exists $\varepsilon > 0$ such that $\mathbb{B}^3_\varepsilon \subseteq V$). Let us introduce $y_0 \in k^*$ such that $0 < |y_0| < \frac{\varepsilon}{r}$. In particular $\frac{|y_0|}{\varepsilon} = \left|\frac{y_0}{\mu}\right| < \frac{1}{r}$. Then $X' = \{(x,y,z) \in \mathbb{B}^3_\varepsilon \mid y = y_0\}$ is a Zariski-closed subset of $\mathbb{B}^3_\varepsilon$, isomorphic to a bidisc $\mathbb{B}^2$. Now, $S'' := S \cap X'$ is defined by

$$S'' = \{(x,z) \in \mathbb{B}^2_\varepsilon \mid \left|\frac{x}{y_0}\right| \leq 1 \text{ and } z = y_0 f(\frac{x}{y_0})\}.$$

As we said, X' is isomorphic to $\mathbb{B}^2$ with coordinates (x', z') where $x = \mu x'$ and $z = \mu z'$. In these new coordinates, $S'' = \{(x',z') \in \mathbb{B}^2 \mid \left|\frac{x'\mu}{y_0}\right| \leq 1 \text{ and } z'\mu = y_0 f(\frac{x'\mu}{y_0})\}$. If we define $g(x') = \frac{y_0}{\mu} f(\frac{x'\mu}{y_0})$ and $s = \frac{|y_0|}{\varepsilon} = \left|\frac{y_0}{\mu}\right| < \frac{1}{r}$, then g has a radius of convergence which is exactly ρ where $s < \rho = \left|\frac{y_0}{\mu}\right| r < 1$, and $\|g\| < \|f\| < 1$, so $S'' = \{(x',z') \in \mathbb{B}^2 \mid |x'| \leq s \text{ and } z' = g(x')\}$, should be semianalytic, but is not (see proposition 1.2.6). □

From this one can deduce:

Corollary 1.2.11. *Let X be a strictly k-analytic space which contains a closed ball of dimension ≥ 3. Then there are overconvergent subanalytic subsets of X which are not rigid semianalytic. In particular, the class of overconvergent subanalytic subsets of X properly contains the class of locally semianalytic subsets of X.*

In conclusion, in figure 1.1, we have shown inequalities $1, 4, 5$ and 8. Now $2, 3, 6, 7$ are set-theoretical consequences of 4, 5 and of the inclusions from the left to the right.

1.2.4 Berkovich points versus rigid points

Let $X = \mathcal{M}(\mathcal{A})$ be a strictly k-affinoid space. We denote by X_{rig} the set of rigid points of X. When one deals with semianalytic or overconvergent subanalytic subsets S of X, one can wonder if things change if we restrict to $S_{\text{rig}} = S \cap X_{\text{rig}}$. Actually the following two propositions show that there is no difference if one works with Berkovich spaces or rigid spaces.

To be precise, let us denote by $\mathcal{B}$ be the free boolean algebra whose set of variables consists in the set of *formal inequalities* $\{|f| \leq |g|\}$, $\{|f| < |g|\}$ and $\{f = 0\}$, for $f, g \in \mathcal{A}$. We denote by SA_{rig} the class of semianalytic subsets of X_{rig} and by SA_{Ber} the class of semianalytic subsets of the Berkovich space X. Then we define natural applications $\alpha : \mathcal{B} \to SA_{\text{Ber}}$ and $\beta : \mathcal{B} \to SA_{\text{rig}}$ where for instance $\alpha(\{|f| \leq |g|\}) = \{x \in X \mid |f(x)| \leq |g(x)|\}$ and $\beta(\{|f| \leq |g|\}) = \{x \in X_{\text{rig}} \mid |f(x)| \leq |g(x)|\}$. In addition we consider the forgetful map $\iota : SA_{\mathrm{Ber}} \to SA_{\mathrm{rig}}$: if $S \in SA_{\mathrm{Ber}}$ is a semianalytic set, $\iota(S) = S \cap X_{\mathrm{rig}}$. We then obtain the commutative diagram:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\alpha} & SA_{\mathrm{Ber}} \\ & \searrow^{\beta} & \downarrow \iota \\ & & SA_{\mathrm{rig}} \end{array}$$

Proposition 1.2.12. *The map ι is bijective.*

Proof. First, ι is surjective by definition.

Now if $\iota(S_1) = \iota(S_2)$, we must show that $S_1 = S_2$. Considering $S_1 \setminus S_2$ and $S_2 \setminus S_1$, we can restrict to show that if $S \in SA_{\mathrm{Ber}}$ and $\iota(S) = \emptyset$, then $S = \emptyset$. According to what has been previously done, we can assume that $S \in SA_{\mathrm{Ber}}$ is a finite intersection of subsets of the form $\{x \in X \mid |f(x)| \le |g(x)| \ne 0\}$, $\{x \in X \mid |f(x)| < |g(x)|\}$ and $\{x \in X \mid h(x) = 0\}$, and that $\iota(S) = S \cap X_{\mathrm{rig}} = \emptyset$. Passing to $Y = \mathcal{M}(\mathcal{A}/\mathcal{I})$ where $\mathcal{I}$ is the ideal generated by the functions h appearing in the third case ($h(x) = 0$), we can assume that S is a finite intersection of subsets of the form: $\{x \in X \mid |f(x)| \le |g(x)| \ne 0\}$ and $\{x \in X \mid |f(x)| < |g(x)|\}$. But then it forms a non empty strictly analytic domain of X so $S \cap X_{\mathrm{rig}} \ne \emptyset$. □

If we denote by CD the family of finite subsets of constructible data of X, by OC the family of overconvergent constructible subsets of X, and OC_{rig} the family of subsets of X_{rig} which are the intersection of an element of OC with X_{rig}, then we can define as above the following commutative diagram:

$$\begin{array}{ccc} CD & \xrightarrow{\alpha} & OC \\ & \searrow^{\beta} & \downarrow \iota \\ & & OC_{\mathrm{rig}} \end{array}$$

To be precise, if $\mathcal{D} \in CD$ is the set of the constructible data $(X_i, T_i) \overset{\varphi_i}{\dashrightarrow} X$, then

$$\alpha(\mathcal{D}) = \bigcup_{i=1}^{n} \varphi_i(T_i).$$

Proposition 1.2.13. *In the above digram, ι is a bijection.*

Proof. Since we showed that OC (and OC_{rig}) is stable under complementary, we can restrict to show that if $S \in OC$ is such that $\iota(S) = S \cap X_{\mathrm{rig}} = \emptyset$, then $S = \emptyset$. To show this we can even assume that $S = \varphi(T)$, where $(Y, T) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum. But, if T is a non empty semianalytic subset of Y, according to proposition 1.2.12, $T_{\mathrm{rig}} \ne \emptyset$, so since φ preserves the rigid points, $\varphi(T)_{\mathrm{rig}} = S_{\mathrm{rig}}$ is non empty.

□

1.3 Overconvergent subanalytic sets when $\dim(X) = 2$

In this section, k will be non-Archimedean algebraically closed field. In this context, if X is a k-analytic space, it is equivalent to say that X is regular, or that X is **quasi-smooth** [Duc, section 3]. We will rather use the second terminology.

The aim of this section is to give a proof of theorem 1.3.12 which asserts that if $\dim(X) = 2$ and X is regular, overconvergent subanalytic sets correspond to locally semianalytic sets. In higher dimension, this result is false. For instance, the example given in proposition 1.2.10 shows that when $n \geq 3$, there exist some overconvergent subanalytic sets of $\mathbb{B}^n$ which are not rigid semianalytic, hence neither locally semianalytic.

We have included in this section for completeness, but most of the ideas contained in it can be found in [Sch94b] and [LR96].

1.3.1 Algebraization of functions

Proposition 1.3.1. *Let X, Y be two k-affinoid spaces, so that we can consider the cartesian diagram*

$$\begin{array}{ccc} & X \times Y & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & & Y \end{array}$$

Let $z \in X \times Y$, and let us denote by $z_1 = \pi_1(z)$, $z_2 = \pi_2(z)$. Let us assume that $z_2 \in Y(k) = Y_{rig}$.

(a) Let V be an affinoid domain of $X \times Y$ such that $z \in V$. There exists an affinoid domain U of X (which contains z_1) such that if W is an affinoid neighbourhood of z_2 small enough, $V \cap (X \times W) = U \times W$.

(b) Let $\mathcal{V}$ be a neighbourhood of z. There exists U (resp. W) an affinoid neighbourhood of z_1 (resp. z_2) such that $\mathcal{V} \supseteq U \times W$

Proof. (a) [Sch94b, 2.2] Let us set $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$. First, using the Gerritzen-Grauert theorem, we can assume that V is a rational domain of $X \times Y$ defined by:

$$V = \{x \in X \times Y \mid |f_i(x)| \leq |g(x)|,\ i = 1 \dots n \text{ and } |g(x)| \geq r\}$$

where $f_i, g \in \mathcal{A}\widehat{\otimes}_k\mathcal{B}$, and $r > 0$. Since we assume that $z_2 \in Y(k)$, it has a sense to evaluate the functions f_i, g in z_2, and we will denote by $f_{i_{z_2}}$, g_{z_2} the corresponding functions, that we see as elements of $\mathcal{A}$ and of $\mathcal{A}\widehat{\otimes}_k\mathcal{B}$. In addition, since z_2 is a rigid point of Y, there exists an affinoid neighbourhood T of z_2 in Y such that

$$\forall i \sup_{x \in X \times T} |(f_i - f_{i_{z_2}})(x)| < r \tag{1.32}$$

$$\sup_{x \in X \times T} |(g - g_{z_2})(x)| < r. \tag{1.33}$$

Since $g = g_{z_2} + (g - g_{z_2})$, we conclude from (1.33) that if $x \in X \times T$,

$$|g(x)| \geq r \Leftrightarrow |g_{z_2}(x)| \geq r. \tag{1.34}$$

Then since also $f_i = f_{i_{z_2}} + (f_i - f_{i_{z_2}})$, from (1.32), (1.33) and (1.34), we conclude that if $x \in X \times T$

$$\big(|g(x)| \geq r \text{ and } |f_i(x) \leq |g(x)|\big) \Leftrightarrow \big(|g_{z_2}(x)| \geq r \text{ and } |f_{i_{z_2}}(x)| \leq |g_{z_2}(x)|\big).$$

Hence, if we set

$$U = \{x \in X \mid |(f_i)_{z_2}(x)| \leq |g_{z_2}(x)| \text{ and } |g_{z_2}(x)| \geq r\},$$

then $V \cap (X \times T) = U \times T$. It then follows that if W is an affinoid domain of Y such that $W \subset T$, then $V \cap (X \times W) = U \times W$.

(b) We can assume that $\mathcal{V} = V$ is an affinoid neighbourhood of z. In (a), $V \cap (X \times W)$ is still a neighbourhood of z, since W is an affinoid neighbourhood of z_2 (because z_2 is a rigid point). If we denote by $s_{z_2} : X \to X \times Y$ the section of π_1 defined by $s_{z_2}(t) = (t, z_2)$, then

$$s_{z_2}^{-1}((V \cap (X \times W)) = s_{z_2}^{-1}(U \times W) = U$$

is an affinoid neighbourhood of x (since $s_{z_2}(x) = z$). Thus U is also an affinoid neighbourhood of z_1.

□

Remark 1.3.2. Without the assumption that $z_2 \in Y(k)$ the previous corollary would be false. Indeed assume that k is algebraically closed. Take for instance $X = \mathcal{M}(k\{x\})$ and $Y = \mathcal{M}(k\{y\})$, and let $\varphi : \mathcal{M}(k\{t\}) \to X \times Y$ be defined by $\varphi(t) = (t, -t)$. Let η be the Gauss point of $\mathcal{M}(k\{t\})$ and $z := \varphi(\eta)$. Let $V = \{p \in \mathcal{M}(k\{x, y\}) \mid |(x + y)(p)| \leq \frac{1}{2}\}$. It is a neighbourhood of z. However, $\pi_1(z)$ (resp $\pi_2(z)$) is the Gauss point $z_1 = \eta_X$ of $\mathcal{M}(k\{x\})$ (resp. $z_2 = \eta_Y$ the Gauss point of $\mathcal{M}(k\{y\})$). It is then easy to see, according to the description of an affinoid domain of the unit disc as a Swizz cheese, that there does not exist an affinoid neighbourhood U (resp. W) of η_X (resp. η_Y) such that $V \supseteq U \times W$. For instance for the reason that in U there would necessarily exist a rigid point $x_0 \in \{x \in k \mid |x| \leq 1\}$ such that $\overline{x_0} = \overline{0}$ and in W a rigid point y_0 such that $\overline{y_0} = \overline{1}$ but $(x_0, y_0) \notin V$ (where $\overline{x}$ corresponds to the reduction of x in $\tilde{k}$).

Lemma 1.3.3. *Let $x \in X = \mathcal{M}(\mathcal{A})$, and let $f = \sum_{n \in \mathbb{N}} a_n T^n \in \mathcal{A}\{r^{-1}T\}$. Let us assume that $f_x \neq 0$. Then there exist $V = \mathcal{M}(\mathcal{B})$ an affinoid domain of X which contains x, $P \in \mathcal{B}[T]$, and $u \in \mathcal{B}\{r^{-1}T\}$ a multiplicative unit such that $f_{|V \times \mathbb{B}_r} = uP$.*

Proof. Since $f_x = \sum_{n \in \mathbb{N}} a_n(x) T^n \neq 0$, this series is distinguished of some order $s \geq 0$ for some $s \geq 0$. We remind that this means that $|a_s(x)| r^s = \|f_x\|$ and that s is the greatest rank for this property.

We now use lemme 1.1.31 in our specific situation where the polyradius $\underline{r}$ is in fact the real number r. Hence we can introduce a finite subset $J \subseteq \mathbb{N}$ such that $s \in J$ and some series $\phi_n \in \mathcal{A}\{r^{-1}T\}$ for $n \in J$ satisfying $\|\phi_n\| < 1$ such that $f = \sum_{n \in J} a_n(T^n + \phi_n)$.

We then define V as the rational domain:

$$V = \{z \in X \mid |a_s(z)| = |a_s(x)| \text{ and } |a_i(z)| r^i \leq |a_s(x)| r^s \text{ for } i \in J \setminus \{s\}\}$$

and denote by $\mathcal{B}$ the affinoid algebra of V. It is then true that $x \in V$. Moreover, on $V = \mathcal{M}(\mathcal{B})$, one checks that a_s is a multiplicative unit, and it follows that in $\mathcal{B}\{r^{-1}T\}$, f is distinguished of order s. One can then apply Weierstrass preparation (corollary 1.1.30) to conclude. □

Remark 1.3.4. The previous result (lemma 1.3.3) is false if we remove the assumption $f_x \neq 0$.

Indeed, let us consider a real number r satisfying $0 < r < 1$, and let $f \in k\{r^{-1}x\}$ be a function whose radius of convergence is exactly r and let us assume that $\|f\| < 1$. Let then $\mathcal{A} = k\{y, t\}$, $X = \mathcal{M}(\mathcal{A})$ the unit bidisc, p the rigid point of X corresponding to the origin, and let us consider

$$F(y, t, x) = y - tf(x) \in k\{y, t\}\{r^{-1}x\} = \mathcal{A}\{r^{-1}X\}.$$

Then we claim that there does not exist $V = \mathcal{M}(\mathcal{B})$ an affinoid domain of X containing p such that $F_{|V \times \mathbb{B}_r} = uP$ where u is a multiplicative unit of $\mathcal{B}\{r^{-1}T\}$ and $P \in \mathcal{B}[t]$.

Proof. Indeed otherwise, there would exist some closed bidisc V of radius $s = |\lambda| \in |k^\times|$ where $\lambda \in k^*$, and some $P \in k\{s^{-1}y, s^{-1}t\}[x]$ and a multiplicative unit $u \in k\{s^{-1}y, s^{-1}t\}\{r^{-1}x\}$ such that

$$F_{|V\times\mathbb{B}_r} = uP. \tag{1.35}$$

Let us fix $t = \lambda$. Then we consider

$$G(y, x) = F(y, \lambda, x) = y - \lambda f(x) \in k\{y, r^{-1}x\}.$$

According to (1.35), $G_{|\mathbb{B}_s\times\mathbb{B}_r} = u(y, \lambda, t)P(y, \lambda, t)$. Replacing y by $\frac{y}{\lambda}$ and f by λf, we then obtain that

$$G(y, x) = y - f(x) \in k\{y, r^{-1}x\}$$
$$G = uP$$

where $u \in k\{y, r^{-1}x\}$ is a multiplicative unit, $P \in k\{y\}[x]$ and $\|f\| < 1$ has a radius of convergence exactly $r < 1$. This implies that if we set

$$S := \{(x, y) \in \mathbb{B}^2 \mid |x| \leq r \text{ and } y = f(x)\}$$

then

$$S = \{(x, y) \in \mathbb{B}^2 \mid |x| \leq r \text{ and } P(x, y) = 0\}$$

so S would be semianalytic in $\mathbb{B}^2$, but in section 1.2, we exploited many times that this is not the case. □

Lemma 1.3.5 (Local algebraization of a function in a family of rings)**.** *Let n be an integer and let us consider $a_0, \dots, a_n$ some elements of $\{x \in k \mid |x| \leq 1\}$ and $r_0, \dots, r_n$ some positive real numbers. Let $Y \subseteq \mathcal{M}(k\{T\}) = \mathbb{B}$ be the Laurent domain defined by*

$$Y = \{y \in \mathcal{M}(k\{T\}) \mid |(T - a_0)(y)| \leq r_0 \text{ and } |(T - a_i)(y)| \geq r_i,\ i = 1 \dots n\},$$

and let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space. Let

$$f \in \mathcal{O}(X \times Y)$$

and let

$$z \in X \times Y$$

such that $\pi_1(z) = x \in X(k)$ and let us set $y := \pi_2(z)$. Assume that $f_x \in \mathcal{H}(x) \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y)$ is non-zero[4]*. Then there exist $V = \mathcal{M}(\mathcal{B})$ an affinoid neighbourhood of x, and $Y' \subset Y$ defined by*

$$Y' = \{y \in \mathcal{M}(k\{T\}) \mid |(T - b_0)(y)| \leq s_0 \text{ and } |(T - b_i)(y)| \geq s_i,\ i = 1 \dots m\}$$

an affinoid neighbourhood of y such that

$$f_{|V\times Y'} = (uP)_{|V\times Y'}$$

where the s_i's are positive real numbers, $b_i \in k^\circ$, u is a multiplicative unit of $V \times Y'$ and $P \in \mathcal{B}[T, (T - b_1)^{-1}, \dots, (T - b_m)^{-1}]$.

Remark 1.3.6. Let us mention that in the proof we distinguish two very different cases.

1. If y is a rigid point then Y' can in fact be chosen to be a closed ball, i.e. $m = 0$.

4. Here $\mathcal{H}(x) \simeq k$ because $x \in X(k)$.

2. Otherwise, if y is not a rigid point, then we can take $s_0 = r_0$, that is to say, we do not have to decrease the radius of the ambient closed ball, but in counterpart, we possibly have to remove some open balls.

Proof. If y is a rigid point, we can indeed find a closed disc Y' which contains y and the result follows from lemma 1.3.3.

If y is not a rigid point, $f_x \in \mathcal{H}(x) \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y)$. Then according to classical results on factorization of functions on rational domains of the closed disc (cf [FvdP04, 2.2.9]), there exist $\alpha_1, \dots, \alpha_N \in k$, $d_1, \dots, d_N \in \mathbb{N}$, g an invertible function of $\mathcal{O}(Y)$ such that

$$f_x = \prod_{i=1}^{N} (T - \alpha_i)^{d_i} g. \tag{1.36}$$

We then set $m = n + N$, $b_i = a_i$ and $s_i = r_i$ for $i = 0 \dots n$, and $b_{n+j} = \alpha_j$ for $j = 1 \dots N$ and we take s_{n+j} small enough so that

$$\{z \in Y \mid |T - \alpha_j|(z) \geq s_{n+j},\ j = 1 \dots N\}$$

is a neighbourhood of y (this is possible because y is not a rigid point). Then we (re)-define

$$Y' := \{y \in \mathcal{M}(k\{T\}) \mid |(T - b_0)(y)| \leq s_0 \textit{ and } |(T - b_i)(y)| \geq s_i, i = 1 \dots m\}.$$

Next, we set

$$G := f \prod_{i=1}^{N} (T - \alpha_i)^{-d_i} \in \mathcal{O}(X \times Y').$$

Then, according to (1.36) $G_x = g$ which does not vanish on Y'_x. So there exists an affinoid neighbourhood $V = \mathcal{M}(\mathcal{B})$ of x such that G is invertible on $V \times Y'$ because the locus of points x where G_x is invertible is open. Now using the explicit description of $\mathcal{O}(V \times Y')$, we can write

$$G = \sum_{\nu = (\nu_0, \dots, \nu_n) \in \mathbb{N}^{m+1}} b_\nu (T - b_0)^{\nu_0} (T - b_1)^{-\nu_1} \dots (T - b_m)^{-\nu_m}.$$

Now for $M \geq 0$ set

$$G_M = \sum_{|\nu| \leq M} b_\nu (T - b_0)^{\nu_0} (T - b_1)^{-\nu_1} \dots (T - b_m)^{-\nu_m}.$$

By definition, $G_M \in \mathcal{B}[T, (T - b_1)^{-1}, \dots, (T - b_m)^{-1}]$. In addition,

$$G_M \xrightarrow[M \to \infty]{} G,$$

so G_M is invertible for M big enough. For such a M,

$$G = G_M + (G - G_M) = G_M(1 + G_M^{-1}(G - G_M)). \tag{1.37}$$

Moreover, if we take M again larger, we can assume that $\|G_M^{-1}\| = \|G^{-1}\|$, and as a consequence

$$\|G_M^{-1}(G - G_M)\| \xrightarrow[M \to \infty]{} 0.$$

Thus, for M large enough, if we set

$$u_M = 1 + G_M^{-1}(G - G_M)$$

then u_M is a multiplicative unit, and according to (1.37)

$$f = G_M u_M \prod_{i=1}^{N} (T - \alpha_i)^{d_i}.$$

We then set $u := u_M$ and $P := G_M \prod_{i=1}^{N} (T - \alpha_i)^{d_i}$ to conclude. □

1.3.2 Blowing up

From now on, X will be a quasi-smooth k-analytic space of dimension 2.

We now make two simple remarks that we will use in the proof of theorem 1.3.12.

Lemma 1.3.7. *Let $\mathcal{A}$ be a k-affinoid algebra, $X = \mathcal{M}(\mathcal{A})$, $0 < r < s$ some real numbers and $h \in \mathcal{A}$.*

1. *Consider the Weierstrass domain of X:*

$$V = \{x \in X \mid |h(x)| \leq s\}$$

and let S be a locally semi-analytic subset of V such that

$$S \subseteq \{x \in X \mid |h(x)| \leq r\}.$$

Then S is a also a locally semianalytic subset of X.

2. *Consider the Laurent domain of X:*

$$V = \{x \in X \mid |h(x)| \geq r\}$$

and let S be a locally semi-analytic subset of V such that

$$S \subseteq \{x \in X \mid |h(x)| \geq s\}.$$

Then S is a also a locally semianalytic subset of X.

Proof. Choose a real number t such that $r < t < s$.

1. Let us set $W = \{x \in X \mid t \leq |h(x)|\}$. Then $\{V, W\}$ is a wide covering of X, and $S \cap V$ is by hypothesis locally semianalytic in V, and by assumption, $S \cap W = \emptyset$ so is also locally semianalytic in W, hence S is locally semianalytic in X.
2. Likewise, let us set $W = \{x \in X \mid |h(x)| \leq t\}$. Then $\{V, W\}$ is a wide covering of X, $S \cap V$ is locally semianalytic in V and $S \cap W = \emptyset$, so S is locally semianalytic in X.

□

This lemma will be used jointly with the following remark:

Remark 1.3.8. Let us consider a k-affinoid space $X = \mathcal{M}(\mathcal{A})$, $f, g \in \mathcal{A}$, $0 < s < r$ and

$$(Z, S) \xrightarrow{\varphi} X$$

the elementary constructible datum given by $Z = \mathcal{M}(\mathcal{B})$ where $\mathcal{B} = \mathcal{A}\{r^{-1}t\}/(f - tg)$ and

$$S = \{z \in Z \mid |f(z)| \leq s|g(z)| \neq 0\}.$$

Moreover, let

$$(Y, U) \overset{\psi}{\dashrightarrow} (Z, S)$$

be a constructible datum.

A. Let us assume that $g|f$. In other words, there exists $h \in \mathcal{A}$ such that $f = gh$. Let us then consider $\mathcal{C} = \mathcal{A}\{r^{-1}t\}/(h - t)$ and $V = \mathcal{M}(\mathcal{C})$. Note that V is the Weierstrass domain of X defined by

$$V = \{x \in X \mid |h(x)| \leq r\}.$$

Let us denote by β the map of the immersion of the affinoid domain V inside X, and let

$$T = \{x \in V \mid |h(x)| \leq s \text{ and } g(x) \neq 0\}.$$

Since $f - tg = g(h - t)$, $(h - t)|(f - tg)$, and there is a closed immersion $V \xrightarrow{\alpha} Z$. Moreover, $\alpha(T) = S$.

Indeed $\alpha(T) \subseteq S$, follows from their respective definitions. Conversely, if $z \in S$, $(f - tg)(z) = 0 = g(z)(h - t)(z)$ but since $g(z) \neq 0$, $(h - t)(z) = 0$ which implies that $z \in V$, and by the definition of S, it follows that $z \in \alpha(T)$.

Let us then consider the following cartesian diagram of k-germs:

$$\begin{array}{ccccc} (Y,U) & \overset{\psi}{\dashrightarrow} & (Z,S) & \overset{\varphi}{\longrightarrow} & X \\ {\scriptstyle \alpha'}\uparrow & & {\scriptstyle \alpha}\uparrow & \nearrow {\scriptstyle \beta} & \\ (Y',U') & \overset{\psi'}{\dashrightarrow} & (V,T) & & \end{array}$$

Here, $(Y',U') \overset{\psi'}{\dashrightarrow} (V,T)$ is still a constructible datum according to corollary 1.1.12. Since $\alpha(T) = S$, it follows that $\alpha(\psi'(U'))) = \psi(U)$, so

$$\varphi(\psi(U)) = \varphi(\alpha(\psi'(U'))) = \beta(\psi'(U')). \tag{1.38}$$

Roughly speaking, we were starting with the constructible datum

$$(Y,U) \overset{\psi}{\dashrightarrow} (Z,S) \xrightarrow{\varphi} X$$

such that the elementary constructible datum of φ was defined with functions f and g such that $g|f$. And we have been able to replace φ by the constructible datum

$$(Y',U') \overset{\psi'}{\dashrightarrow} (V,T) \xrightarrow{\beta} X$$

where V is a Weierstrass domain. Note moreover that T and so also $\psi'(U')$ satisfy the hypothesis of lemma 1.3.7 (1).

B. If $f|g$, there exists $h \in \mathcal{A}$ such that $g = fh$. Let then $\mathcal{C} = \mathcal{A}\{r^{-1}t\}/(1-th)$, $V = \mathcal{M}(\mathcal{C})$. Note that V is the Laurent domain of X defined by

$$V = \{x \in X \mid |h(x)| \geq \frac{1}{r}\}.$$

Let us denote by β the map of the immersion of the Laurent domain V inside X, and let

$$T = \{x \in V \mid |h(x)| \geq \frac{1}{s} \text{ and } g(x) \neq 0\}.$$

Since $(1 - th)|(f - tg)$, there is a closed immersion $V \xrightarrow{\alpha} Z$. Moreover, $\alpha(T) = S$.

We then consider the following cartesian diagram of k-germs:

$$\begin{array}{ccccc} (Y,U) & \overset{\psi}{\dashrightarrow} & (Z,S) & \overset{\varphi}{\longrightarrow} & X \\ {\scriptstyle \alpha'}\uparrow & & {\scriptstyle \alpha}\uparrow & \nearrow {\scriptstyle \beta} & \\ (Y',U') & \overset{\psi'}{\dashrightarrow} & (V,T) & & \end{array}$$

Here, $(Y', U') \overset{\psi'}{\dashrightarrow} (V, T)$ is still a constructible datum. Since $\alpha(T) = S$, it follows that $\alpha(\psi'(U'))) = \psi(U)$, so

$$\varphi(\psi(U)) = \varphi(\alpha(\psi'(U'))) = \beta(\psi'(U')). \tag{1.39}$$

In that case, we were starting with the constructible datum $(Y, U) \overset{\psi}{\dashrightarrow} (Z, S) \xrightarrow{\varphi} X$ such that $f|g$, and we have been able to replace it by the following constructible datum $(Y', U') \overset{\psi'}{\dashrightarrow} (V, T) \xrightarrow{\beta} X$ where V is a Laurent domain of X. Note moreover that T and so also $\psi'(U')$ satisfies the hypothesis of lemma 1.3.7 (2).

Remark 1.3.9. We are going to use some blowing up of k-analytic spaces in the following context: X will be a quasi-smooth k-analytic space of dimension 2, and we will blow up a rigid point p of X. In particular, the resulting blowing up $\tilde{X}$ will be still quasi-smooth. To give a precise description of the situation, since k is algebraically closed, we can assume that $X = \mathbb{B}^2$ and p is the origin. The blowing up can then be described with two charts as follows. We consider

$$\begin{array}{rcl|rcl} X_1 = \mathcal{M}(k\{x, t_1\}) & \xrightarrow{\pi_1} & \mathbb{B}^2 = \mathcal{M}(k\{x, y\}) & X_2 = \mathcal{M}(k\{y, t_2\}) & \xrightarrow{\pi_2} & \mathbb{B}^2 = \mathcal{M}(k\{x, y\}) \\ (x, t_1) & \mapsto & (x, t_1 x) & (y, t_2) & \mapsto & (t_2 y, y) \end{array}$$

Then $\widetilde{\mathbb{B}^2}$ is obtained by gluing X_1 and X_2 along the domains $U_1 = \{z \in X_1 \mid t_1(z) \neq 0\}$ and $U_2 = \{z \in X_2 \mid t_2(z) \neq 0\}$ via the isomorphism

$$\begin{array}{ccc} U_1 & \to & U_2 \\ (x, t_1) & \mapsto & (x t_1, t_1^{-1}). \end{array}$$

Proposition 1.3.10. *Let $X = \mathcal{M}(\mathcal{A})$ be a quasi-smooth k-affinoid space of dimension 2 and let $f, g \in \mathcal{A}$. Then there exists a succession of blowing up of rigid points $\pi : \tilde{X} \to X$ such that for all $x \in \tilde{X}$, $f_x | g_x$ or $g_x | f_x$. Remark that $\tilde{X}$ is still quasi-smooth.*

Proof. We may assume that X is irreducible. If $f = 0$ or $g = 0$, there is nothing to prove, so we may assume that $f \neq 0$ and $g \neq 0$. Likewise, if $f = g$, there is nothing to do, so we may also assume that $f - g \neq 0$

Let $h = fg(f - g)$. Hence, $h \neq 0$. We can find a succession of blowing up of rigid points $\pi : \tilde{X} \to X$ such that $\pi^*(h)$ is a normal crossing divisor. Indeed, the classical proof (see [Kol07, 1.8]) that works in the algebraic case, or the complex analytic case, can be translated *verbatim* in our context, and since we are dealing with a compact space, the local procedure of [Kol07, 1.8] needs only to be applied to a finite number of points. Let then $x \in \tilde{X}$.

If x is not a rigid point, $\mathcal{O}_{\tilde{X},x}$ is a field or a discrete valuation ring and the result is clear.

Otherwise, if x is a rigid point, its local ring is a regular local ring of dimension 2. By assumption, $h = fg(f - g)$ is a normal crossing divisor, thus can be written in $\mathcal{O}_{\tilde{X},x}$ as

$$(fg(f - g))_x = u \xi_1^n \xi_2^m \tag{1.40}$$

where ξ_1, ξ_2 is a system of local parameters around x and u is a unit in $\mathcal{O}_{\tilde{X},x}$. Dividing by the common divisor of f_x and g_x in $\mathcal{O}_{\tilde{X},x}$, we can assume for instance that $f_x = v \xi_1^p$ and $g_x = w \xi_2^q$ and $f_x - g_x = z \xi_1^a \xi_2^b$ where v, w and z are units of $\mathcal{O}_{\tilde{X},x}$.

If $p > 0$ then modulo ξ_1 we obtain $f = 0$, so $f_x - g_x = w \xi_2^q$ modulo ξ_1. This implies that $a = 0$ and that $b = q$. So $f_x = (f_x - g_x) + g_x$ is divisible by ξ_2^q, and this implies that $q = 0$. So g_x is invertible and, $g_x | f_x$.

And if $p = 0$, then f_x is invertible, so $f_x | g_x$. ☐

Lemma 1.3.11. *Let X be a good quasi-smooth strictly k-analytic space of dimension 2.*

1. *Let $q \in X_{rig}$ and $\pi : \tilde{X} \to X$ the blowing-up of X at q, and let $S \subseteq \tilde{X}$ be a locally semianalytic subset. Then $\pi(S)$ is locally semianalytic.*
2. *If $\pi : \tilde{X} \to X$ is a succession of blowing-up of rigid points, and $S \subseteq \tilde{X}$ is locally semianalytic, then $\pi(S)$ is also locally semianalytic.*

Proof. (2) is a consequence of (1) so we only have to show (1).

The problem is local on X, and since outside q, π is a local isomorphism, we can restrict to an affinoid neighbourhood of q, and since X is regular at q, we can assume that $X = \mathbb{B}^2$ and q is the origin.

Then $\pi : \tilde{X} \to X$ can be described with two charts, one of them being

$$\begin{array}{rrcl} \pi_1 : & X_1 = \mathcal{M}(k\{x,t\}) & \to & X = \mathcal{M}(k\{x,y\}) \\ & (x,t) & \mapsto & (x,tx) \end{array}$$

The other chart being analogous we only consider π_1. Now, changing S in $S \cap X_1$, we can restrict to show that if S is locally semianalytic in X_1, so is $\pi_1(S)$. Since π_1 induces an isomorphism between $X_1 \setminus V(x)$ and $\{p \in \mathbb{B}^2 \mid |y(p)| \leq |x(p)| \neq 0\}$, we only have to show that $\pi_1(S)$ is semianalytic around q, the origin of $\mathbb{B}^2$.

Now if for each $p \in= V(x) \subseteq X_1$ we can find U_p an affinoid neighbourhood of p, and $\varepsilon_p > 0$ such that $\pi_1(U_p \cap S) \cap \mathbb{B}^2_{\varepsilon_p}$ is semianalytic in $\mathbb{B}^2_{\varepsilon_p} \subseteq X$, then by compactness of $V(x)$, we can extract $U_1, \ldots, U_n$ a finite covering of $V(x)$ and $\varepsilon > 0$ such that

$$\bigcup_{i=1}^{n} (\pi_1(U_i \cap S)) \cap \mathbb{B}^2_\varepsilon = \pi_1(S) \cap \mathbb{B}^2_\varepsilon$$

is semianalytic in $\mathbb{B}^2_\varepsilon$.

So to conclude the proof, it is enough to fix $p \in V(x)$ and to find U an affinoid neighbourhood of p, and $\varepsilon > 0$ such that $\pi_1(U \cap S) \cap \mathbb{B}^2_\varepsilon$ is semianalytic in $\mathbb{B}^2_\varepsilon$.

Since S is locally semianalytic in X_1, we can find U an affinoid neighbourhood of p such that $U \cap S$ is semianalytic in U. According to corollary 1.3.1, we can assume that [5] $U = \mathbb{B}_\varepsilon \times W$ where

$$W = \{w \in \mathcal{M}(k\{t\}) \mid |(t - a_0)(w)| \leq r_0 \text{ and } |(t - a_i)(w)| \geq r_i, \ i = 1 \ldots n\}$$

for some $a_0, \ldots, a_n \in k^\circ$ and $r_0, \ldots, r_n \in \mathbb{R}_+$.

We can also assume that S is of the following form:

$$S = \bigcap_{j=1}^{m} \{v \in U \mid |f_j(v)| \bowtie_j |g_j(v)|\}.$$

Since $U = \mathbb{B}_\varepsilon \times W$ with $\mathbb{B}_\varepsilon = \mathcal{M}(k\{\varepsilon^{-1}x\})$, we can factor each f_j and g_j by the greatest power of x which divides them, hence introduce some integers b_j, c_j such that

$$S = \bigcap_{j=1}^{m} \{v \in U \mid |x^{b_j} \tilde{f}_j(v)| \bowtie_j |x^{c_j} \tilde{g}_j(v)|\}$$

where the series $\tilde{f}_j(0,t)$ and $\tilde{g}_j(0,t)$ are non zero, and $f_j = x^{b_j} \tilde{f}_j, g_j = x^{c_j} \tilde{g}_j$. To simplify the notations, we will use f_j (resp. g_j) instead of $\tilde{f}_j$ (resp. $\tilde{g}_j$), so that

$$S = \bigcap_{j=1}^{m} \{v \in U \mid |x^{b_j} f_j(v)| \bowtie_j |x^{c_j} g_j(v)|\}$$

5. Here we use the explicit description of affinoid domains of $\mathbb{B}$.

where the series $f_j(0,t)$ and $g_j(0,t)$ are non zero.

Then according to lemma 1.3.5 we can decrease ε and W so that for $j = 1,\dots,m$ $f_j = u_j F_j$ (resp. $g_j = v_j G_j$) where u_j (resp. v_j) is a multiplicative unit, and F_j (resp. G_j) $\in k\{\varepsilon^{-1}x\}[t,(t-a_1)^{-1},\dots,(t-a_n)^{-1}]$.

Said differently, there exist an integer N, for $j = 1,\dots,m$ some $P_j, Q_j \in k\{\varepsilon^{-1}x\}[t]$ such that

$$f_j = u_j.\frac{P_j}{((t-a_1)\dots(t-a_n))^N}$$
$$g_j = v_j.\frac{Q_j}{((t-a_1)\dots(t-a_n))^N}$$

where u_j and v_j are multiplicative units. Hence if $v \in U$,

$$|f_j(v)| \bowtie_j |g_j(v)| \Leftrightarrow \left|u_j(v)\frac{P_j(v)}{((t-a_1)\dots(t-a_n))^N(v)}\right| \bowtie_j \left|v_j(v)\frac{Q_j(v)}{((t-a_1)\dots(t-a_n))^N(v)}\right|$$
$$\Leftrightarrow |P_j(v)| \bowtie_j \lambda_j |Q_j(v)|$$

where

$$\lambda_j = \frac{\|v_j\|}{\|u_j\|} \in |k^\times|.$$

Moreover,

$$S \cap U = (S \cap \{v \in U \mid x(v) = 0\}) \cup (S \cap \{v \in U \mid x(v) \neq 0\})$$

and $\pi_1(\{v \in U \mid x(v) = 0\}) = q$, the origin of $\mathbb{B}^2$.

So, adding if necessary the origin to $\pi_1(S \cap \{v \in U \mid v(x) \neq 0\})$ (which will not change the fact that it is semianalytic), we can restrict to show that $\pi_1(S \cap \{v \in U \mid v(x) \neq 0\})$ is semianalytic around the origin. Moreover since on $\{v \in U \mid v(x) \neq 0\}$, π_1 is bijective, the following holds:

$$\pi_1\left(\bigcap_{j=1}^m \{v \in U \mid |x^{b_j} f_j(v)| \bowtie_j |x^{c_j} g_j(v)|\} \cap \{v \in U \mid x(v) \neq 0\}\right)$$
$$= \bigcap_{j=1}^m \pi_1\left(\{v \in U \mid |x^{b_j} f_j(v)| \bowtie_j |x^{c_j} g_j(v)|\}\right) \cap \{v \in U \mid x(v) \neq 0\}.$$

Now since $y = tx$ and $P_j \in k\{\varepsilon^{-1}x\}[t]$ there exists an integer $M \geq 0$ such that $x^M P_j(x,t) \in k\{\varepsilon^{-1}x\}[tx] = k\{\varepsilon^{-1}x\}[y]$, i.e. $x^M P_j(x,t) = \pi^*(\tilde{P}_j(x,y))$ for some $\tilde{P}_j(x,y) \in k\{\varepsilon^{-1}x\}[y]$ and such that $x^M Q_j(x,t) \in k\{\varepsilon^{-1}x\}[y]$, i.e. $x^M Q_j(x,t) = \pi^*(\tilde{Q}_j(x,y))$ for some $\tilde{Q}_j(x,y) \in k\{\varepsilon^{-1}x\}[y]$.

Now if $v \in U$ and $x(v) \neq 0$ the following holds:

$$|x^{b_j} f_j(v)| \bowtie_j |x^{c_j} g_j(v)| \Leftrightarrow |x^{M+b_j} f_j(v)| \bowtie_j |x^{M+c_j} g_j(v)|$$
$$\Leftrightarrow |x^{b_j} \tilde{P}_j(\pi_1(v))| \bowtie_j \lambda_j |x^{c_j} \tilde{Q}_j(\pi_1(v))|.$$

From that we conclude that

$$z \in \pi_1\left(\bigcap_{j=1}^m \{v \in U \mid |x^{b_j} f_j(v)| \bowtie_j |x^{c_j} g_j(v)|\} \cap \{v \in U \mid x(v) \neq 0\}\right)$$

$$\Leftrightarrow z \in \bigcap_{j=1}^{m} \{z \in \pi_1(U) \mid |x^{b_j}\tilde{P}_j(z)| \bowtie_j |x^{c_j}\tilde{Q}_j(z)|\} \cap \{z \in X \mid x(z) \neq 0\}.$$

Since $\pi_1(U)$ is semianalytic in $\mathbb{B}^2_\varepsilon$, we conclude that $\pi_1(S \cap \{v \in U \mid v(x) \neq 0\})$ is semianalytic in $\mathbb{B}^2_\varepsilon$, which ends the proof. □

Theorem 1.3.12. *Let X be a good quasi-smooth strictly k-analytic space of dimension 2 with k algebraically closed, and $S \subseteq X$. Then S is overconvergent subanalytic subset if and only if S is locally semianalytic.*

Proof. Since the problem is local, we can assume that X is affinoid and that $S = \varphi(U)$ where $(Y,U) \overset{\varphi}{\dashrightarrow} X$ is a constructible datum, and just check that S is locally semianalytic. We do it by induction on the complexity of φ. So let $(Y,U) \overset{\varphi}{\dashrightarrow} X$ be a constructible datum, that we decompose as

$$\varphi = (Y,U) \overset{\psi}{\dashrightarrow} Z \overset{\chi}{\to} X$$

where χ is an elementary constructible datum, and ψ a constructible datum whose complexity is one less than φ. We can introduce $f, g \in \mathcal{A}$, $0 < s < r$ such that $Z = \mathcal{M}(\mathcal{A}\{r^{-1}t\}/(f - tg))$. According to proposition 1.3.10, we can find a succession of blowing-up of rigid points $\pi : \tilde{X} \to X$ such that for all $x \in \tilde{X}$, $f_x | g_x$ or $g_x | f_x$. According to remark 1.3.9, $\tilde{X}$ is still quasi-smooth. This gives us the following cartesian diagram:

$$\begin{array}{ccc} (Y,U) & \overset{\varphi}{\dashrightarrow} & X \\ {\scriptstyle \pi'}\uparrow & & \uparrow{\scriptstyle \pi} \\ (Y',U') & \overset{\varphi'}{\dashrightarrow} & \tilde{X} \end{array}$$

Then $\varphi(U) = \pi(\varphi'(U'))$. Moreover, since $\tilde{X}$ is compact, we can then find a finite wide covering $\{X_i\}_{i=1}^n$ of $\tilde{X}$ by affinoid domains such that for all i, $(f_{|X_i})|(g_{|X_i})$ or $(g_{|X_i})|(f_{|X_i})$. We denote by $\pi_i : X_i \to X$ the composition of the embedding of the affinoid domain $X_i \to \tilde{X}$ with $\pi : \tilde{X} \to X$. This gives the following cartesian diagrams:

$$\begin{array}{ccccc} (Y,U) & \overset{\psi}{\dashrightarrow} & Z & \overset{\chi}{\longrightarrow} & X \\ {\scriptstyle \pi_i''}\uparrow & & \uparrow{\scriptstyle \pi_i'} & & \uparrow{\scriptstyle \pi_i} \\ (Y_i,U_i) & \overset{\psi_i}{\dashrightarrow} & Z_i & \overset{\chi_i}{\longrightarrow} & X_i \end{array}$$

Then

$$\varphi(U) = \pi(\varphi'(U')) = \pi\left(\bigcup_{i=1}^{n} \chi_i(\psi_i(U_i))\right)$$

But $(Y_i, U_i) \overset{\psi_i}{\dashrightarrow} Z_i$ is a constructible datum of lower complexity than φ, so that we would like to use our induction hypothesis, and claim that $\psi_i(U_i)$ is locally semianalytic. However, Z_i is not necessarily still quasi-smooth so we cannot do that. Yet, since $(f_{|X_i})|(g_{|X_i})$ or $(g_{|X_i})|(f_{|X_i})$, according to remark 1.3.8, we can in fact replace Z_i by a Weierstrass (or a Laurent) domain of X_i, and hence assume that Z_i is quasi-smooth. Thus by induction hypothesis $\psi_i(U_i)$ is locally semianalytic in Z_i.

Next we use lemma 1.3.7 to assert that $\chi_i(\psi_i(U_i))$ is locally semianalytic in X_i. So

$$\varphi'(U') = \bigcup_{i=1}^{n} (\chi_i(\psi_i(U_i))$$

is locally semianalytic in $\tilde{X}$, since $\{X_i\}$ was a wide covering of $\tilde{X}$. Finally, according to lemma 1.3.11, $\pi(\varphi'(U')) = S$ is locally semianalytic. □

Chapter 2

Cohomology of semialgebraic sets

We fix a prime number $\ell \neq \mathrm{char}(\tilde{k})$.

Introduction

Let $\mathcal{X}$ a separated k-scheme of finite type. One can define ℓ-adic cohomology groups $H_c^i(\mathcal{X}^{an}, \mathbb{Q}_\ell)$ which have good properties (in particular, they are finite dimensional vector spaces when k is algebraically closed).

If U is a semi-algebraic subset of $\mathcal{X}^{an}$, using the theory of k-germs developed in [Ber93], it is possible to define cohomology groups with compact support of the k-germ $(\mathcal{X}^{an}, U)$, that we will denote by $H_c^i(U, \mathbb{Q}_\ell)$. We want to point out that in general, U is not equipped with a structure of k-analytic space.

In this chapter, we generalize the finiteness property mentioned above to locally closed semi-algebraic subsets of $\mathcal{X}^{an}$. More precisely, let $\widehat{k^a}$ be the completion of the algebraic closure of k and let us set $\overline{\mathcal{X}} := \mathcal{X}^{an}\widehat{\otimes}_k\widehat{k^a}$ and $\pi : \overline{\mathcal{X}} \to \mathcal{X}^{an}$ the natural morphism. If U is a subset of $\mathcal{X}^{an}$, we set $\overline{U} := \pi^{-1}(U)$. Our main result is:

Theorem. ***2.6.18*** *Let $\mathcal{X}$ be a separated k-scheme of finite type of dimension d, U a locally closed semi-algebraic subset of $\mathcal{X}^{an}$, and $\ell \neq$char$(\tilde{k})$ be a prime number.*

1. *The groups $H_c^i(\overline{U}, \mathbb{Q}_\ell)$ are finite dimensional $\mathbb{Q}_\ell$-vector spaces, on which $Gal(k^{sep}/k)$ acts continuously, and $H_c^i(\overline{U}, \mathbb{Q}_\ell) = 0$ for $i > 2d$.*
2. *If $V \subset U$ is a semi-algebraic subset which is open in U and $F = U \setminus V$, then there is a long exact sequence of Galois modules*

$$\cdots \to H_c^i(\overline{V}, \mathbb{Q}_\ell) \longrightarrow H_c^i(\overline{U}, \mathbb{Q}_\ell) \longrightarrow H_c^i(\overline{F}, \mathbb{Q}_\ell) \longrightarrow H_c^{i+1}(\overline{V}, \mathbb{Q}_\ell) \to \cdots$$

3. *For all integer n there are canonical isomorphisms of Galois modules:*

$$\bigoplus_{i+j=n} H_c^i\left(\overline{U}, \mathbb{Q}_\ell\right) \otimes H_c^j\left(\overline{V}, \mathbb{Q}_\ell\right) \simeq H_c^n\left(\overline{U \times V}, \mathbb{Q}_\ell\right)$$

We prove more generally this result when $\mathcal{X}$ is a separated $\mathcal{A}$-scheme of finite type where $\mathcal{A}$ is a k-affinoid algebra. The above result corresponds to the case $\mathcal{A} = k$.

This question was raised by F. Loeser and used in [HL11] where they study the Milnor fibration associated to a morphism $f : X \to \mathbb{A}^1_{\mathbb{C}}$ where X is a smooth complex algebraic variety. The non-Archimedean field is then $k = \mathbb{C}((t))$, and they use the analytic Milnor fiber introduced in [NS07].

In fact, the main point in the above theorem is to prove that when k is algebraically closed, the groups $H^i_c(\overline{U}, \mathbb{Z}/\ell^n\mathbb{Z})$ are finite. If X is a compact k-analytic space, we say that a subset $S \subset X$ is *rigid semianalytic* if there exists a finite covering $\{X_i\}$ of X by some affinoid domains such that for each i, $S \cap X_i$ is semianalytic in X_i. We in fact prove:

Proposition. ***2.3.1*** *Let us assume that k is algebraically closed, and let X be a compact k-analytic space. Then for any locally closed rigid semianalytic subset S of X and Λ a finite group whose cardinal is prime to char($\tilde{k}$), the groups $H^q_c((X,S),\Lambda)$ are finite.*

We want to point out that this result relies deeply on the cohomological finiteness of affinoid spaces which has been recently proved by V. Berkovich in [Ber13].

Finally in section 2.5, we explain some counterparts of these finiteness results for Huber's adic spaces.

2.1 Reminder on k-germs

We fix X a Hausdorff k-analytic space, and $S \subset X$ a locally closed subset of X.

The étale site of a germ

(see [Ber93, 3.4]) If S is a subset of X, (X,S) is called a k**-germ**. If (Y,T) is another k-germ, a morphism of k-germs $f : (Y,T) \to (X,S)$ is a morphism of k-analytic spaces $f : Y \to X$ such that $f(T) \subset S$. This defines the category of k-germs.

Then the category of k-$\mathscr{G}$erms is defined as the localization of the category of k-germs by the morphisms of k-germs $\varphi : (Y,T) \to (X,S)$ which induce an isomorphism of Y with some open neighbourhood of S in X (this implies that φ induces a homeomorphism between T and S). It is important to remark that the functor k-$\mathscr{A}$n $\to$ k-$\mathscr{G}$erms defined by $X \to (X, |X|)$ is fully faithful.

A morphism of k-$\mathscr{G}$erms f is called étale if it has a representative $\varphi : (Y,T) \to (X,S)$ such that $\varphi : Y \to X$ is étale and $T = \varphi^{-1}(S)$. Berkovich defines the small étale site $(X,S)_{\text{ét}}$ of the k-germ (X,S), as the category of étale morphisms above (X,S), a family $(Y_i,T_i) \xrightarrow{f_i} (Y,T)$ being a covering if $\cup_i f_i(T_i) = T$. The category of abelian sheaves on $(X,S)_{\text{ét}}$ is denoted by $\boldsymbol{S}(X,S)_{\text{ét}}$.

Cohomology groups with compact support

(see [Ber93, 5.1]) If S is a topological space, a family of supports Φ is a family of closed subsets of S which is stable under finite unions and such that if F is a closed subset of S and $F \subset T$ for some $T \in \Phi$ then $F \in \Phi$. If Φ is a family of supports of S, and $A \subset S$ we set

$$\Phi_A = \{F \in \Phi \mid F \subset A\} \tag{2.1}$$

which is a family of supports of A.

A family of supports Φ is said to be paracompactifying if for all $A \in \Phi$, A is paracompact, and if for all $A \in \Phi$, there exists $B \in \Phi$ which is a neighbourhood of A.

If Φ is a family of supports, the following functor

$$\begin{array}{rccl} \Gamma_\Phi : & \boldsymbol{S}(X,S)_{\text{ét}} & \to & \text{Ab} \\ & F & \mapsto & \{s \in F(X,S) \mid \text{supp}(s) \in \Phi\} \end{array} \tag{2.2}$$

is left exact. Its right derived functors are denoted by $H^q_\Phi((X,S),F)$.

Let us assume that S is locally closed in X, that T is an open subset of S, and that $R := S \setminus T$. If Φ is a paracompactifying family of supports of S, and F is an abelian sheaf on (X, S), there is a long exact sequence [Ber93, 5.2.6 (ii)]:

$$\cdots \to H^{q-1}_{\Phi_R}((X,R), F_{(X,R)}) \to H^q_{\Phi_T}((X,T), F_{(X,T)}) \to H^q_{\Phi}((X,S), F) \to H^q_{\Phi_R}((X,R), F_{(X,R)}) \to \cdots . \tag{2.3}$$

Now, we denote by C_S the family of compact subsets of S. If S is Hausdorff, this is a family of supports, and if S is locally compact, C_S is paracompactifying. Remind that if S is a locally closed subset of a locally compact topological set X, S is also locally compact. Since we will always consider locally closed subsets S of some Hausdorff k-analytic space X, the family of supports C_S will then be paracompactifying. If $F \in \boldsymbol{S}(X,S)_{\text{ét}}$, we will denote by

$$H^q_c((X,S), F) := H^q_{C_S}((X,S), F)$$

the associated right derived functors.

Let $T \subset S$ be an open subset of S and $R := S \setminus T$ the complementary closed subset of S. Then, $(C_S)_T = C_T$ because being compact in T or in S is equivalent; likewise $(C_S)_R = C_R$. In this context, the long exact sequence (2.3) can be written:

$$\cdots \to H^{q-1}_c((X,R), F_{(X,R)}) \to H^q_c((X,T), F_{(X,T)}) \to H^q_c((X,S), F) \to H^q_c((X,R), F_{(X,R)}) \to \cdots \tag{2.4}$$

What we will look at

Let Λ be a finite abelian group whose cardinal is prime to the characteristic of $\tilde{k}$. We set

$$H^n_c(S, \Lambda) := H^n_c((X,S), \underline{\Lambda}) \tag{2.5}$$

where $\underline{\Lambda}$ is the constant sheaf of value Λ on $(X,S)_{ét}$. This notation is abusive since the cohomology of S itself is meaningless, only the cohomology of the k-germ (X,S) can be defined. Nonetheless, we will use the notation (2.5) to simplify the exposition.

If we still denote by $\underline{\Lambda}$ the constant sheaf of value Λ on $(X,S)_{ét}$, then if $U \subset S$, $\underline{\Lambda}_{(X,U)}$ is isomorphic to the constant sheaf of value Λ on $(X,U)_{ét}$. Hence, if T is an open subset of S and $R := S \setminus T$, the long exact sequence (2.4) becomes

$$\cdots \to H^{q-1}_c(R, \Lambda) \to H^q_c(T, \Lambda) \to H^q_c(S, \Lambda) \to H^q_c(R, \Lambda) \to \cdots \tag{2.6}$$

Quasi-immersions (see [Ber93, 4.3])

A morphism of k-germs $\varphi : (Y,T) \to (X,S)$ is called a **quasi-immersion** if φ induces a homeomorphism of T with its image $\varphi(T)$ and for all $y \in T$, if we set $x := \varphi(y)$, the maximal purely inseparable extension of $\mathcal{H}(x)$ in $\mathcal{H}(y)$ is everywhere dense in $\mathcal{H}(y)$.

Here are two examples of quasi-immersions that we will use frequently. If U is an analytic domain of X, and $\varphi : U \to X$ is the natural inclusion morphism, then $\varphi : (U,U) \to (X,U)$ is a quasi-immersion. If $\varphi : Z \to X$ is a closed immersion, then $(Z,Z) \to (X,Z)$ is a quasi-immersion. Moreover, quasi-immersions are stable under composition and base change.

Quasi-immersions will be very important for us through the following result:

Proposition. *[Ber93, 4.3.4 (i)] If $\varphi : (Y,T) \to (X,S)$ is a quasi-immersion of k-germs, it induces an equivalence of categories*

$$\boldsymbol{S}(Y,T)_{ét} \simeq \boldsymbol{S}(X, \varphi(T))_{ét}.$$

In particular if U is an analytic domain of X, there are isomorphisms

$$H_c^q(U,\Lambda) \simeq H_c^q((U,U),\Lambda) \simeq H_c^q((X,U),\Lambda). \tag{2.7}$$

Similarly, if Z is a closed k-analytic subset of X,

$$H_c^q(Z,\Lambda) \simeq H_c^q((Z,Z),\Lambda) \simeq H_c^q((X,Z),\Lambda). \tag{2.8}$$

We want to stress out that (2.7) and (2.8) partly justify the abuse of the notation made in (2.5).

2.2 A finiteness result in the affinoid case

In this section k will be a (complete) non-Archimedean algebraically closed field. We consider a k-affinoid algebra $\mathcal{A}$, and we set $X = \mathcal{M}(\mathcal{A})$. We remind that Λ is a finite abelian group whose order is prime to the characteristic of $\tilde{k}$. The goal of this section is to prove proposition 2.2.3.

Lemma 2.2.1. *Let $n \in \mathbb{N}$, and for $i = 1 \dots n$, let $f_i, g_i \in \mathcal{A}$, $\bowtie_i \in \{<, \leq\}$ and $\lambda_i > 0$ be a positive real number. Let us consider*

$$S = \bigcap_{i=1}^{n} \{x \in X \mid |f_i(x)| \bowtie_i \lambda_i |g_i(x)|\}.$$

Then, the groups $H_c^q(S,\Lambda)$ are finite for all $q \in \mathbb{N}$.

Proof. We prove the lemma by induction on n.

If $n = 0$, then $S = X$ and the result is a consequence of the finiteness result [Ber13, Theorem 1.1].

Let $n \geq 0$ and assume that the result is true for n. Let $f, g \in \mathcal{A}$, $\lambda > 0$, and let

$$\begin{aligned} S =& \bigcap_{i=1}^{n} \left\{x \in X \mid |f_i(x)| \bowtie_i \lambda_i |g_i(x)|,\ i = 1 \dots n\right\} \\ T =& \{x \in S \mid |g(x)|\lambda < |f(x)|\} \\ R =& \{x \in S \mid |f(x)| \leq \lambda |g(x)|\} = S \setminus T. \end{aligned}$$

Let us show that the groups $H_c^q(T,\Lambda)$ and $H_c^q(R,\Lambda)$ are finite. This will achieve our induction step.

By its definition, S is a locally closed subset of X, T is an open subset of S, and $R = S \setminus T$ is the complementary closed subset of S. So we can apply the long exact sequence (2.6) to S, R and T. By induction hypothesis, the groups $H_c^q(S,\Lambda)$ are finite, so if we show that the groups $H_c^q(R,\Lambda)$ are finite, this will also prove the finiteness of the groups $H_c^q(T,\Lambda)$. Let us then show that the groups $H_c^q(R,\Lambda)$ are finite.

Let $Y = \mathcal{M}\left(\mathcal{A}\{\lambda^{-1}U\}/(f - Ug)\right)$ and let $\varphi : Y \to X$ be the morphism of affinoid spaces induced by the natural map $\mathcal{A} \to \mathcal{A}\{\lambda^{-1}U\}/(f - Ug)$. The morphism φ induces an isomorphism between the analytic domain of Y:

$$A = \{y \in Y \mid g(y) \neq 0\}$$

and the analytic domain of X:

$$B = \{x \in X \mid |f(x)| \leq \lambda |g(x)| \text{ and } g(x) \neq 0\}.$$

As a consequence, φ induces a quasi-immersion $\varphi : (Y, A) \to (X, B)$, and also a quasi-immersion

$$(Y, A \cap \varphi^{-1}(S)) \to (X, B \cap S). \tag{2.9}$$

But

$$\varphi^{-1}(S) = \bigcap_{i=1}^{n} \{y \in Y \mid |f_i(y)| \bowtie_i \lambda_i |g_i(y)|\}.$$

Here we have written f_i (resp. g_i) whereas we should rather have written $\varphi^*(f_i)$ (resp. $\varphi^*(g_i)$). Hence by induction hypothesis, the groups $H_c^q((Y, \varphi^{-1}(S)), \Lambda)$ are finite.

Now $A \cap \varphi^{-1}(S)$ is an open subset of $\varphi^{-1}(S)$, whose complement in $\varphi^{-1}(S)$ is $\varphi^{-1}(S) \cap \{y \in Y \mid g(y) = 0\}$. Let Z be the Zariski closed subset of Y defined by

$$Z = \{y \in Y \mid g(y) = 0\}$$

and $\psi : Z \to Y$ the associated closed immersion. We then obtain a quasi-immersion:

$$(Z, \psi^{-1}(\varphi^{-1}(S))) \to (Y, Z \cap \varphi^{-1}(S)).$$

By the induction hypothesis the groups $H_c^q(\psi^{-1}(\varphi^{-1}(S)), \Lambda)$ are finite, therefore it is also true for the groups $H_c^q(Z \cap \varphi^{-1}(S), \Lambda)$. Thus in the long exact sequence

$$\cdots \to H_c^q(A \cap \varphi^{-1}(S), \Lambda) \to H_c^q(\varphi^{-1}(S), \Lambda) \to H_c^q(Z \cap \varphi^{-1}(S), \Lambda) \to \cdots$$

the written groups in the middle and in the right are finite and we conclude from this that the groups $H_c^q(A \cap \varphi^{-1}(S), \Lambda)$ are finite. We have already noticed that $(Y, A \cap \varphi^{-1}(S)) \to (X, B \cap S)$ is a quasi-immersion, hence

$$H_c^q(A \cap \varphi^{-1}(S), \Lambda) \simeq H_c^q(B \cap S, \Lambda).$$

From this, we conclude that the groups $H_c^q(B \cap S, \Lambda)$ are also finite.

If we go back to our starting point

$$B \cap S = \{x \in S \mid |f(x)| \leq \lambda |g(x)| \text{ and } g(x) \neq 0\}$$

is an open subset of

$$R = \{x \in S \mid |f(x)| \leq \lambda |g(x)|\}.$$

The complementary subset of $B \cap S$ in R is

$$D = \{x \in S \mid |f(x)| \leq \lambda |g(x)| \text{ and } g(x) = 0\} = \{x \in S \mid f(x) = g(x) = 0\}.$$

We denote by Z' the Zariski closed subset of X :

$$Z' = \{x \in X \mid f(x) = g(x) = 0\}$$

hence $D = Z' \cap S$, and using the same kind of arguments as above we can conclude that the groups $H_c^q(D, \Lambda)$ are finite.

We use for the last time the long exact sequence

$$\cdots \to H_c^q(B \cap S, \Lambda) \to H_c^q(R, \Lambda) \to H_c^q(D, \Lambda) \to \cdots$$

We have shown that the groups on the left, and on the right are finite, thus the groups $H_c^q(R, \Lambda)$ are also finite. □

Next, we want to extend this result to an arbitrary locally closed semianalytic subset of X. In order to do so, we introduce the following notation.

Let $f_1, \dots f_r, g_1, \dots g_r \in \mathcal{A}$, and $\lambda_1, \dots, \lambda_r > 0$. For a subset $I \subseteq \{1 \dots r\}$ we set

$$C_I = \left(\bigcap_{i \in I} \{x \in X \mid |f_i(x)| \leq \lambda_i |g_i(x)|\} \right) \cap \left(\bigcap_{j \notin I} \{x \in X \mid |f_j(x)| > \lambda_j |g_j(x)|\} \right).$$

The subsets C_I induce a partition of X, and each C_I is a semianalytic set of X. If $A \subseteq \mathcal{P}(\{1 \dots r\})$, let us set

$$C_A = \coprod_{I \in A} C_I.$$

This is a semianalytic subset of X, and in fact every semianalytic subset of X is of this form [1]. This follows from the fact that if S is a semianalytic subset of X, one can find some $f_1, \dots, f_r, g_1, \dots, g_r \in \mathcal{A}$ such that S is a finite union of subsets of the form

$$\{x \in \ \mid |f_{i_1}(x)| \bowtie_{i_1} |g_{i_1}(x)| \text{ and } \cdots \text{ and } |f_{i_m}(x)| \bowtie_{i_m} |g_{i_m}(x)|\}$$

where $1 \leq i_1 < \ldots < i_m \leq r$, and $\bowtie_j \in \{\leq, >\}$.

For instance, if $S = \{|f_1| \leq |g_1|\} \cup \{|f_2| > |g_2|\}$, $A = \{\{1,2\}, \{1\}, \emptyset\}$ is suitable:

$$S = \{|f_1| \leq |g_1| \text{ and } |f_2| \leq |g_2|\} \cup \{|f_1| \leq |g_1| \text{ and } |f_2| > |g_2|\} \cup \{|f_1| > |g_1| \text{ and } |f_2| > |g_2|\}.$$

Lemma 2.2.2. *Let r and n be two integers, $f_1, \dots f_r, g_1, \dots g_r, F_1, \dots F_n, G_1, \dots G_n \in \mathcal{A}$, $A \subseteq \mathcal{P}(\{1 \dots r\})$, $\bowtie_i \in \{<, \leq\}$ for $i = 1 \dots n$ and $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_n$ be some positive real numbers. Let us suppose that the semianalytic set of X*

$$C = C_A \cap \Big(\bigcap_{i=1}^{n} \{x \in X \mid |F_i(x)| \bowtie_i \mu_i |G_i(x)|\} \Big)$$

is locally closed. Then the groups $H^q_c(C, \Lambda)$ are finite.

Proof. We prove this by induction on r.

If $r = 0$ this is precisely the preceding lemma 2.2.1.

Let $r \geq 0$ and let us assume that we are given $f_1, \dots f_{r+1}, g_1, \dots g_{r+1} \in \mathcal{A}$, $A \subset \mathcal{P}(\{1 \dots r+1\})$, and

$$C = C_A \cap \left(\bigcap_{i=1}^{n} \{x \in X \mid |F_i(x)| \bowtie_i \mu_i |G_i(x)|\} \right)$$

a subset of X, assumed to be locally closed. Then we must show that the groups $H^q_c(C, \Lambda)$ are finite. The idea is to decompose C as

$$C = \{x \in C \mid |f_{r+1}(x)| \leq \lambda_{r+1} |g_{r+1}(x)|\} \coprod \{x \in C \mid |f_{r+1}(x)| > \lambda_{r+1} |g_{r+1}(x)|\}$$

and to use our induction hypothesis to this partition of C.

To formalize this, we set

$$A_1 = \{P \in A \mid r+1 \in P\}$$

$$A_2 = \{P \in A \mid r+1 \notin P\} = A \setminus A_1.$$

1. This is some kind of disjunctive normal form.

Finally we set

$$B_1 = \{P \setminus \{r+1\} \mid P \in A_1\}$$

and $B_2 = A_2$. In addition, we see B_1 and B_2 as subsets of $\mathcal{P}(\{1 \ldots r\})$.

We now consider the subsets of X, C_{B_1} and C_{B_2}, associated with $f_1, \ldots f_r, g_1, \ldots g_r$ and $\lambda_1, \ldots, \lambda_r$. Then, by definition of B_1 and B_2,

$$C_A = (\{x \in X \mid |f_{r+1}(x)| \leq \lambda_{r+1}|g_{r+1}(x)|\} \cap C_{B_1}) \coprod (\{x \in X \mid |f_{r+1}(x)| > \lambda_{r+1}|g_{r+1}(x)|\} \cap C_{B_2}).$$

Said more simply, we have partitioned the set $C_A = \coprod_{I \in A} C_I$ in two parts: on the left side, we have kept the $C_I's$ where the inequality $|f_{r+1}(x)| \leq \lambda_{r+1}|g_{r+1}(x)|$ appears, and on the right side, we have kept the $C_I's$ where the inequality $|f_{r+1}(x)| > \lambda_{r+1}|g_{r+1}(x)|$ appears, which allows us to restrict to subsets of $\{1 \ldots r\}$. And now we set:

$$C_1 = C_{B_1} \cap \left(\{x \in X \mid |f_{r+1}(x)| \leq \lambda_{r+1}|g_{r+1}(x)|\} \cap \bigcap_{i=1}^{n} \{x \in X \mid |F_i(x)| \bowtie_i \mu_i |G_i(x)|\} \right)$$
$$C_2 = C_{B_2} \cap \left(\{x \in X \mid |f_{r+1}(x)| > \lambda_{r+1}|g_{r+1}(x)|\} \cap \bigcap_{i=1}^{n} \{x \in X \mid |F_i(x)| \bowtie_i \mu_i |G_i(x)|\} \right).$$

The following holds :

$$C_1 = \{x \in C \mid |f_{r+1}(x)| \leq \lambda_{r+1}|g_{r+1}(x)|\}$$

$$C_2 = \{x \in C \mid |f_{r+1}(x)| > \lambda_{r+1}|g_{r+1}(x)|\}.$$

So $C = C_1 \coprod C_2$, C_2 is an open subset of C, and C_1 is the closed complementary subset attached to it, in particular, C_1 and C_2 are locally closed in X. But we can now apply our induction hypothesis to C_1 and C_2: the groups $H^q_c(C_i, \Lambda)$ are finite for $i = 1, 2$. Finally, according to long exact sequence (2.6) applied to $C_2 \subset C \supset C_1$, the groups $H^q_c(C, \Lambda)$ are finite. □

The previous lemma, with $n = 0$ becomes:

Proposition 2.2.3. *Let S be a locally closed semianalytic subset of X. The groups $H^q_c(S, \Lambda)$ are finite.*

2.3 Global results

In this section, we will still assume that k is algebraically closed.

2.3.1 The compact case

Proposition 2.3.1. *Let X be a compact k-analytic space, and let S be a locally closed G-semianalytic subset of X. The groups $H^q_c(S, \Lambda)$ are finite.*

Proof. We prove by induction on m that if X is a Hausdorff k-analytic space which is covered by m affinoid domains: $X = \cup_{i=1}^m V_i$, and $S \subset X$ is a locally closed subset of X such that $S \cap V_i$ is semianalytic in V_i for all i (in particular S is rigid semianalytic in X), then the groups $H^q_c(S, \Lambda)$ are finite.

For $m = 1$ this is proposition 2.2.3.

Let then $m \geq 1$ and let us assume that X is covered by the affinoid domains V_i, $i = 1 \dots m+1$, and that $S \subset X$ such that for all i, $S \cap V_i$ is semianalytic in V_i. We set

$$\begin{aligned} R &= S \cap (V_1 \cup \ldots \cup V_m) \\ T &= S \setminus R \\ &= (V_{m+1} \cap S) \setminus (V_{m+1} \cap (V_1 \cup \ldots \cup V_m)). \end{aligned} \tag{2.10}$$

If $X' := V_1 \cup \ldots V_m$, then X' is a compact analytic domain of X (not necessarily good), and R is a locally closed rigid semianalytic subset of X' such that $R \cap V_i$ is semianalytic in V_i for $i = 1 \dots m$. Thus by induction hypothesis, the groups $H^q_c((X', R), \Lambda)$ are finite. In addition, since $(X', R) \to (X, R)$ is a quasi-immersion (because X' is an analytic domain of X), $H^q_c((X', R), \Lambda) \simeq H^q_c((X, R), \Lambda)$, hence these are finite groups.

Next, we claim that T is a locally closed semianalytic set of V_{m+1}. Indeed, for each i, $V_{m+1} \cap V_i$ is an affinoid domain of V_{m+1} (because X is separated), hence closed and semianalytic in V_{m+1} according to the Gerritzen-Grauert theorem. Hence $V_{m+1} \cap (V_1 \cup \ldots V_n)$ is a closed semianalytic set of V_{m+1}. Since $S \cap V_{m+1}$ is a locally closed semianalytic subset of V_{m+1}, according to (2.10), T is a locally closed semianalytic subset of V_{m+1}. Hence according to proposition 2.2.3, the groups $H^q_c((V_{m+1}, T), \Lambda)$ are finite, and since $(V_{m+1}, T) \to (X, T)$ is a quasi-immersion, $H^q_c((V_{m+1}, T), \Lambda) \simeq H^q_c((X, T), \Lambda)$, thus the groups $H^q_c((X, T), \Lambda)$ are finite.

Finally, since R is a closed subset of S and $T = S \setminus R$, the long exact sequence (2.6) allows to conclude that the groups $H^q_c((X, S), \Lambda)$ are finite. □

2.3.2 The semi-algebraic case

Proposition 2.3.2. *Let $\mathcal{A}$ be a k-affinoid algebra, $\mathcal{X}$ a separated $\mathcal{A}$-scheme of finite type, and S a locally closed semi-algebraic subset of $\mathcal{X}^{an}$. Then the groups $H^q_c((\mathcal{X}^{an}, S), \Lambda)$ (that we abusively denote by $H^q_c(S, \Lambda)$) are finite.*

Proof. According to Nagata's compactification theorem (see [Con07] for a modern proof), we can embed $\mathcal{X}$ as an open subscheme of a proper $\mathcal{A}$-scheme $\overline{\mathcal{X}}$. Since $(\mathcal{X}^{an}, \mathcal{X}^{an}) \to (\overline{\mathcal{X}}^{an}, \mathcal{X}^{an})$ is a quasi-immersion, and since quasi-immersions are stable under base change, for all $q \geq 0$ we have an isomorphism of groups:

$$H^q_c((\mathcal{X}^{an}, S), \Lambda) \simeq H^q_c((\overline{\mathcal{X}}^{an}, S), \Lambda).$$

In addition, according to proposition 0.3.14, S is still semi-algebraic in $\overline{\mathcal{X}}^{an}$. Moreover, S is still locally closed in $\overline{\mathcal{X}}^{an}$ because $\mathcal{X}^{an}$ is open in $\overline{\mathcal{X}}^{an}$. So we can assume that $\mathcal{X}$ is proper.

In that case, $\mathcal{X}^{an}$ is compact, and according to lemma 0.3.17, S is rigid semianalytic in $\overline{\mathcal{X}}^{an}$, and the result follows from proposition 2.3.1.

□

2.4 A finiteness result for overconvergent subanalytic sets

We now want to explain how cohomological finiteness results such as proposition 2.3.2 can be extended to overconvergent subanalytic sets. In this section, k will be algebraically closed, and Λ will still be a finite abelian group whose cardinal is prime to $\mathrm{char}(\tilde{k})$.

Proposition 2.4.1. *Let X be a k-affinoid space, $S \subset X$ a locally closed overconvergent subanalytic subset of X. Then the groups $H^q_c((X, S), \Lambda)$ are finite.*

Proof. Let us denote by $\mathcal{A}$ the k-affinoid algebra of X. According to theorem 1.1.40, we can assume that

$$S = \bigcup_{i=1}^{n} \varphi_i(S_i)$$

where $(X_i, S_i) \overset{\varphi_i}{\dashrightarrow} X, i = 1 \dots n$, are n constructible data.

Let us then denote by c_i the complexity of the constructible datum φ_i. Remind (definition 1.1.5) that c_i corresponds to the number of divisions involved in the definition of φ_i, or equivalently, the number of new variables which have been introduced to define the affinoid space X_i. We then prove the result by induction on

$$c := \sum_{i=1}^{n} c_i.$$

If $c = 0$, this means that for each i, $\varphi_i = id$, so that S is in fact a locally closed semianalytic subset of X and the result follows from proposition 2.2.3.

Let $c > 0$ and let us assume that the result holds for $c' < c$. We can assume that $c_1 > 0$, so that we can write

$$\varphi_1 : (X_1, S_1) \overset{\psi}{\dashrightarrow} (Z, T) \overset{\varphi}{\to} X \tag{2.11}$$

where φ is an elementary constructible datum. Then,

$$complexity(\psi) = complexity(\varphi_1) - 1 < c_1.$$

Moreover, by definition, there exist $f, g \in \mathcal{A}$ and $s \in \sqrt{|k^\times|}$ such that

$$T \subset \{s \in Z \mid |f(z)| \leq s|g(z)| \neq 0\}. \tag{2.12}$$

Let us set

$$T_1 = \{x \in S \mid f(x) = g(x) = 0\}$$
$$T_2 = \{x \in S \mid f(x) \neq 0 \text{ or } g(x) \neq 0\}.$$

Then T_2 is an open subset of S, and T_1 is its closed complement. According to the long exact sequence (2.6), it is enough to prove that the groups $H^q_c((X, T_i), \Lambda)$, for $i = 1, 2$ are finite.

Since $T_1 \cap \varphi_1(S_1) = \emptyset$ (according to (2.12) and the definition of T_1),

$$T_1 = \bigcup_{i=2}^{n} (\varphi_i(S_i)) \cap \{x \in X \mid f(x) = g(x) = 0\}.$$

So if we set $S'_i := S_i \cap \varphi_i^{-1}(\{x \in X \mid f(x) = g(x) = 0\})$ it is easy to see that S'_i is still semianalytic in X_i and that $T_1 = \cup_{i=2}^{n} \varphi_i(S'_i)$. Moreover, since S is locally closed, by definition of T_1 we see that T_1 is also locally closed. So if we apply the induction hypothesis to the family of constructible data

$$\{(X_i, S'_i) \overset{\varphi_i}{\dashrightarrow} X\}_{i=2\dots n}$$

we see that the groups $H^q_c(X, T_1), \Lambda)$ are finite.

It then remains to show that the groups $H^q_c(X, T_2), \Lambda)$ are finite. For this purpose, let us set

$$U_1 = \{x \in T_2 \mid |f(x)| \leq s|g(x)|\} = \{x \in S \mid |f(x) \leq s|g(x)| \neq 0\}$$
$$U_2 = \{x \in T_2 \mid |f(x)| > s|g(x)|\}.$$

Then U_2 is an open subset of T_2 and U_1 is its closed complement, so thanks to the closed-open long exact sequence (2.6) it is sufficient to show that the groups $H^q_c(X, U_i), \Lambda)$, $i = 1, 2$ are finite.

First, by definition, $U_2 \cap \varphi_1(S_1) = \emptyset$. So

$$U_2 = \bigcup_{i=2}^{n} \varphi_i(S_i) \cap \{x \in X \mid |f(X)| \leq s|g(x)| \neq 0\}$$

and a similar argument as above implies by induction hypothesis that the groups $H^q_c((X, U_2), \Lambda)$ are finite.

Eventually, for $i = 2 \ldots n$, we consider the cartesian diagram of k-germs (remind the definition of ϕ in (2.11):

$$\begin{array}{ccc} (Y_i, S'_i) & \xrightarrow{\psi_i} & (X_i, S_i) \\ \downarrow{\scriptstyle \varphi'_i} & & \downarrow{\scriptstyle \varphi_i} \\ (Z, T) & \xrightarrow{\varphi} & X \end{array}$$

Then, S'_i is semianalytic in Y_i, defined by

$$S'_i = \psi_i^{-1}(S_i) \cap \{x \in Y_i \mid |f(x)| \leq s|g(x)| \neq 0\}.$$

Here write simply f in place of $(\varphi'_i)^*(f)$. Then, by construction, if we set

$$S' := \bigcup_{i=2}^{n} \varphi'_i(S'_i) \cup \psi(S_1)$$

we obtain the equality

$$S' = \varphi^{-1}(U_1).$$

Hence S' is a locally closed overconvergent subanalytic set of Z (since by definition, U_1 is locally closed) defined by the overconvergent constructible data

$$\{(Y_i, S'_i) \overset{\varphi'_i}{\dashrightarrow} Z\}_{i=2\ldots n} \cup \{(X_1, S_1) \overset{\psi}{\dashrightarrow} Z\}.$$

So, the complexities of this family of constructible data add to

$$(c_1 - 1) + c_2 + \ldots + c_n = c - 1.$$

By induction hypothesis, it follows the groups $H^q_c((Z, S'), \Lambda)$ are finite. Now $(Z, S') \xrightarrow{\varphi} (X, U_1)$ is a quasi-immersion, and $U_1 = \varphi(S')$ so $H^q_c((Z, S'), \Lambda) \simeq H^q_c((X, U_1), \Lambda)$, and these groups are then finite, which ends the proof. $\square$

2.5 Analogous statements for adic spaces

In this section, k will be a non-Archimedean algebraically closed non-trivially valued field, Λ a finite group prime to the characteristic of $\tilde{k}$ and $\mathcal{A}$ a strictly k-affinoid algebra.

We want to stress out that in the previous sections, instead of working with the étale cohomology developed by Berkovich in [Ber93], we could also have used the theory of adic spaces and its étale cohomology theory, developed by R. Huber in [Hub96]. This might be interesting because this will define different groups (cf. remark 2.5.1) and will apply for different semianalytic (resp. semi-algebraic) subsets (cf. remark 2.5.7). To avoid confusion, we will denote by $H^q_{c,ad}(X,\Lambda)$ the groups defined by the cohomology with compact support of adic spaces.

In this framework, the analogue of a a k-germ (X,S) is the notion of a pseudo-adic space (X,S) over $\mathrm{Spa}(k,k^\circ)$ [*Ibid.*, p. **1**.10.3]. The quasi-immersions will be replaced by locally closed embeddings [*Ibid.*, p. **1**.10.8 (ii)], and the analogue of [Ber93, 4.3.4] which states that cohomology is invariant by quasi-immersion is [Hub96, 2.3.8] which states the same thing for locally closed embeddings. In Huber's theory though, compactly supported cohomology isn't defined as a derived functor, but with some compactification, like in the étale cohomology of schemes. Nonetheless, one can check that if $i:(X,S)\to(Y,T)$ is a locally closed embedding with $i(S)=T$, then $H^q_{c,ad}((X,S),i^*(\mathcal{F}))\simeq H^q_{c,ad}((Y,T),\mathcal{F})$. Indeed, in this case, $i_! = i_*$ is an exact functor (it induces an equivalence of categories), so $R^+i_! = i_! = i_*$ [*Ibid.*, p. **5**.4.1]. So $R^+i_!(i^*\mathcal{F})\simeq\mathcal{F}$, from what it follows that $H^q_{c,ad}((X,S),i^*(\mathcal{F}))\simeq H^q_{c,ad}((Y,T),\mathcal{F})$.

Remark 2.5.1. One has to keep in mind that compactly supported cohomology does not give the same groups in both theories, for instance if X is the closed disc of radius one:

i	0	1	2
$H^i_{c,Ber}(X,\Lambda)$	Λ	0	0
$H^i_{c,ad}(X,\Lambda)$	0	0	Λ

In section 2.2, we systematically used the long exact sequence

$$\cdots H^{q-1}_c(R,\Lambda)\to H^q_c(T,\Lambda)\to H^q_c(S,\Lambda)\to H^q_c(R,\Lambda)\to\cdots$$

where

$$T=\{x\in S \mid |f(x)|<|g(x)|\} \tag{2.13}$$

and $R=S\setminus T$. Although the closed-open long exact sequence is still valid for pseudo-adic spaces [Hub96][5.5.11 (iv)], T as defined in (2.13) is not an open subset of S any more, so we cannot apply this long exact sequence. In fact the typical example of an open subset of an adic space is

$$T=\{x\in S \mid |f(x)|\le|g(x)|\neq 0\}. \tag{2.14}$$

It will be then possible in that case to apply this long exact sequence (which includes the case $\{f\neq 0\}=\{0\le|f|\neq 0\}$).

Remark 2.5.2. A subset S is a finite boolean combination of subsets of the form $\{|f|\le|g|\neq 0\}$ if and only if it is a finite boolean combination of subsets of the form $\{|f|\le|g|\}$.

For instance, $\{|f|\le|g|\}=\{|f|\le|g|\neq 0\}\cup(\{g\neq 0\}\cup\{f\neq 0\})^c$.

Let $\mathcal{A}$ be a (strictly) k-affinoid algebra. We will say that a subset $S\subset\mathrm{Spa}(\mathcal{A},\mathcal{A}^\circ)$ is semianalytic if it is a boolean combination of subsets of the form

$$\{x\in\mathrm{Spa}(\mathcal{A},\mathcal{A}^\circ)\mid |f(x)|\le|g(x)|\}$$

where $f, g \in \mathcal{A}$. This definition slightly differs from the one given for Berkovich spaces: here we do not allow real constants in the inequalities.

Lemma 2.5.3. *Let $X = Spa(\mathcal{A}, \mathcal{A}^\circ)$ be the affinoid adic space associated to $\mathcal{A}$, $S = \cap_{i=1}^n S_i$ where for each i, S_i is of the form $S_i = \{x \in X \mid |f_i(x)| \leq |g_i(x)| \neq 0\}$ or $S_i = \{x \in X \mid |f_i(x)| > |g_i(x)|$ or $g_i(x) = 0\}$, with $f_i, g_i \in \mathcal{A}$. Then the groups $H^q_{c,ad}(S, \Lambda)$ are finite.*

Proof. Mimic the proof of lemma 2.2.1 using that

$$\{x \in X \mid |f_i(x)| \leq |g_i(x)| \neq 0\}^c = \{x \in X \mid |f_i(x)| > |g_i(x)| \text{ or } g_i(x) = 0\}.$$

The key point here (that makes possible the base case of the induction) is that for an affinoid adic space Y, the groups $H^q_{c,ad}(Y, \Lambda)$ are finite [Hub07][5.1]. □

Proposition 2.5.4. *Let T be a locally closed, semianalytic subset of $X = Spa(\mathcal{A}, \mathcal{A}^\circ)$. Then the groups $H^q_{c,ad}(T, \Lambda)$ are finite.*

Proof. According to remark 2.5.2, we can assume that T is a finite union of subsets S as in lemma 2.5.3. Hence we can adapt the proof of lemma 2.2.2. □

In this context, if X is a quasi-separated adic space of finite type over k, we will say that S is *locally semianalytic* if there exists a finite affinoid covering $\{U_i\}$ of X such that $S \cap U_i$ is semianalytic in U_i for all i. Adapting the proofs of proposition 2.3.1, we obtain:

Proposition 2.5.5. *Let X be a quasi-separated adic space of finite type over k, and S a locally closed, locally semianalytic subset of X. Then the groups $H^q_{c,ad}(S, \Lambda)$ are finite.*

We can define similarly semi-algebraic subsets $S \subset \mathcal{X}^{ad}$ where $\mathcal{X}$ is an $\mathcal{A}$-scheme of finite type, like in definition 0.3.8, but without the real constants λ. We then obtain:

Proposition 2.5.6. *Let $\mathcal{X}$ be a separated $\mathcal{A}$-scheme of finite type, S a locally closed semi-algebraic subset of $\mathcal{X}^{ad}$. Then the groups $H^q_{c,ad}(S, \Lambda)$ are finite.*

Remark 2.5.7. As indicated above, if X is a k-analytic (resp. adic) affinoid space, the class of locally closed subspaces will be different according to the theories. To illustrate this we want to give two examples. Let us consider X the closed bidisc of radius 1: $X = \mathcal{M}(k\{x, y\})$ or $\text{Spa}(k\{x, y\}, k^\circ\{x, y\})$ according to theory we are using. Remind that a subset U is locally closed if and only if U is open in $\overline{U}$.

A subset which is locally closed for the topology of adic spaces but not for the Berkovich topology. Let $V = \{p \in X \mid |x(p)| > |y(p)|\} \cup \{p_0\}$. Here p_0 is the rigid point corresponding to the origin. Then V is closed in the adic topology. Indeed its complement is

$$V^c = \{p \in X \setminus \{p_0\} \mid |x(p)| \leq |y(p)|\} = \{p \in X \mid |x(p)| \leq |y(p)| \neq 0\}$$

which is open by definition of the topology of adic spaces. But we claim that V is not locally closed for the Berkovich topology. To show this, for $r, s \leq 1$ let $\eta_{r,s} \in X$ be defined by

$$\eta_{r,s}(\sum_{i,j \in \mathbb{N}} a_{i,j} x^i y^j) = \max_{i,j \in \mathbb{N}} |a_{i,j}| r^i s^j.$$

Then for $r > s$, $\eta_{r,s} \in V$, and for $0 < r \leq 1$, $\eta_{r,r} \in \overline{V} \setminus V$. Now, if V was open in $\overline{V}$, since it contains p_0, it should contain $\eta_{r,r}$ for r small enough which is a contradiction.

A subset which is locally closed for the Berkovich topology but not for the topology of adic spaces. Let us consider the set $U = \{p \in X \mid |x(p)| \leq |y(p)|\}$. Then U

is closed for the Berkovich topology but not locally-closed for the topology of adic spaces. Indeed, if p_0 is the rigid point corresponding to the origin $(0,0)$, $p_0 \in U$, but U is not a neighbourhood of p_0 in $\overline{U}$ with respect to the topology of adic spaces. Otherwise for some $\varepsilon > 0$, U would contain a subset $B = \{p \in \overline{U} \mid |x(p)| \leq \varepsilon \text{ and } |y(p)| \leq \varepsilon\}$. But then for $0 < \alpha < \varepsilon$ with $\alpha \in |k^\times|$, we can define $\eta_\alpha \in \overline{U}$ a valuation of rank 2 such that $\eta_\alpha(x) = \alpha$ and $\eta_\alpha(y) = \alpha_-$ where $\alpha_- < \alpha$ but is infinitesimally closed. Now, $\eta_\alpha \in \overline{U}$ because $\eta_\alpha \in \overline{\{\eta_{\alpha,\alpha}\}}$ (cf. definition above). So by definition of B, $\eta_\alpha \in B$. So we should have $\eta_\alpha \in U$, which is false. So U is not locally closed for the adic topology.

2.6 From torsion to ℓ-adic coefficients

In this section we explain how to pass from $\mathbb{Z}/\ell^n\mathbb{Z}$-coefficients to ℓ-adic coefficients. We want to make an exposition as down to earth as possible. A more general treatment can be found in [Gro77, Exposes V-VI].

2.6.1 Continuous Galois action

From now on, we do not assume any more that k is algebraically closed. We still consider X a Hausdorff k-analytic space. Let S be a locally closed subset of X and let us set $\overline{X} = X\widehat{\otimes}_k\widehat{k^a}$, $\pi : \overline{X} \to X$, the projection, and $\overline{S} = \pi^{-1}(S)$. This is a locally closed subset of $\overline{X}$. There is an action of $\mathrm{Gal}(k^{sep}/k)$ on $\overline{X}$ which stabilizes $\overline{S}$. Hence $\mathrm{Gal}(k^{sep}/k)$ acts on the k-germ $(\overline{X}, \overline{S})$. If $\mathcal{F} \in \boldsymbol{S}(X, S)$, we set $\overline{\mathcal{F}} = \pi^*(\mathcal{F})$. The action of $\mathrm{Gal}(k^{sep}/k)$ on $(\overline{X}, \overline{S})$ induces an action on $H^i_c(\overline{S}, \overline{\mathcal{F}})$. Indeed for $\sigma \in \mathrm{Gal}(k^{sep}/k)$ we have the commutative diagram

$$\begin{array}{ccc} (\overline{X}, \overline{S}) & \xrightarrow{\ \sigma\ } & (\overline{X}, \overline{S}) \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & (X, S) & \end{array}$$

Then the action of σ on the cohomology is given by :

$$\sigma^* :\ H^i_c((\overline{X}, \overline{S}), \overline{\mathcal{F}}) \simeq H^i_c((\overline{X}, \overline{S}), \sigma^*\overline{\mathcal{F}}) \simeq H^i_c((\overline{X}, \overline{S}), \overline{\mathcal{F}}),$$

the last isomorphism being a consequence of the isomorphism $\sigma^* \circ \pi^*(\mathcal{F}) \simeq \pi^*(\mathcal{F})$. If (X, S) is a k-germ, and K is a complete extension of k, we consider $\pi_K : X_K = X\hat{\otimes}_k K \to X$ and we set $S_K = \pi_K^{-1}(S)$, so that we can consider the K-germ (X_K, S_K).

Proposition 2.6.1. *If X is a Hausdorff k-analytic space, F a locally closed subset of X, $\mathcal{F} \in \boldsymbol{S}_{\acute{e}t}(X)$, there is an isomorphism of Galois modules:*

$$\varinjlim_{K/k} H^q_c((X_K, F_K), \mathcal{F}_K) \simeq H^q_c((\overline{X}, \overline{F}), \overline{\mathcal{F}})$$

where the limit is taken over all finite separable extensions K of k contained in k^{sep}.

Proof. We will use that if Y is a Hausdorff k-analytic space and $\mathcal{G} \in \boldsymbol{S}_{\text{ét}}(Y)$, the following is true [Ber93, 5.3.5]:

$$\varinjlim_{K/k} H^q_c(Y_K, \mathcal{G}_K) \simeq H^q_c(\overline{Y}, \overline{\mathcal{G}}),$$

and it is an isomorphism of Galois-modules.

Since F is locally closed, it can be written $F = U \cap F'$ where U is open in X and F' is closed in X, and since $(U, F) \to (X, F)$ is a quasi-immersion, for all $q \geq 0$, $H_c^q((U, F), \Lambda) \simeq H_c^q((X, F), \Lambda)$. Now, F is closed in U, so we can replace X by U and assume that F is closed.

In this situation, let $U = X \setminus F$ be the complementary open subset of X. For K a finite separable extension of k, F_K is a closed subset of X_K whose complementary open subset is U_K. Hence we get a commutative diagram:

$$\begin{array}{ccccccccc}
\cdots\to & \varinjlim_{K/k} H_c^q(U_K, \mathcal{F}_K) & \longrightarrow & \varinjlim_{K/k} H_c^q(X_K, \mathcal{F}_K) & \longrightarrow & \varinjlim_{K/k} H_c^q((X_K, F_K), \mathcal{F}_K) & \to\cdots \\
& \| & & \| & & \downarrow & \\
\cdots\to & H_c^q(\overline{U}, \overline{\mathcal{F}}) & \longrightarrow & H_c^q(\overline{X}, \overline{\mathcal{F}}) & \longrightarrow & H_c^q((\overline{X}, \overline{F}), \overline{\mathcal{F}}) & \to\cdots
\end{array}$$

Thanks to the long exact sequence (2.4), the first row is exact because $\varinjlim$ is an exact functor (we consider a filtered inductive limit), and the second row is exact. We can then conclude thanks to the five lemma. □

In particular, if $\mathcal{A}$ is a k-affinoid algebra, $\mathcal{X}$ is a separated $\mathcal{A}$-scheme of finite type, and if S is a locally closed subset of $\mathcal{X}^{an}$,

$$H_c^q((\overline{\mathcal{X}}, \overline{S}), \Lambda)$$

is a continuous Galois module. Moreover, if T is an open subset of S and $R = S \setminus T$, the long exact sequence

$$\cdots \to H_c^q((\overline{\mathcal{X}}, \overline{T}), \Lambda) \to H_c^q((\overline{\mathcal{X}}, \overline{S}), \Lambda) \to H_c^q((\overline{\mathcal{X}}, \overline{R}), \Lambda) \to \cdots$$

is Galois equivariant.

2.6.2 About the dimension

Let X be a Hausdorff k-analytic space. We denote by d the dimension of X (cf. [Ber90, p. 34] and [Ber93, p. 23]).

Proposition 2.6.2. *[Ber93, Cor 5.3.8]. Let Y be a Hausdorff k-analytic space of dimension d, $\mathcal{F}$ a torsion abelian sheaf on Y, then for all $i > 2d$, $H_c^i(Y, \mathcal{F}) = 0$.*

We can generalize this result in the following way:

Proposition 2.6.3. *Let X be a Hausdorff k-analytic space of dimension d. Let S be a locally closed subset of X. For $q > 2d$, and $F \in \boldsymbol{S}(X)$ an abelian torsion sheaf on X, $H_c^q((X, S), F) = 0$.*

Proof. Write $S = U \cap Z$ with U an open subset of X and Z a closed subset. Set $V = U \setminus S$ which is an open subset of U and X. Then $H_c^q((U, S), F) \simeq H_c^q((X, S), F)$ and $H_c^q(V, F) \simeq H_c^q((V, V), F) \simeq H_c^q((U, V), F)$, hence in the long exact sequence (2.6)

$$\cdots \to H_c^q((U, V), F) \to H_c^q(U, F) \to H_c^q((U, S), F) \to \cdots$$

according to the previous proposition, the groups are 0 on the left and in the middle for $q > 2d$, so this must also occur for the groups on the right. □

In our situation, this result can be refined. If $\mathcal{A}$ is a k-affinoid algebra, $\mathcal{X}$ a separated $\mathcal{A}$-scheme of finite type, and $S \subset \mathcal{X}^{an}$ a semi-algebraic set, we set $Z := \overline{S}^{Zar}$. Then since $(Z, S) \to (X, S)$ is a quasi-immersion, $H_c^q((Z, S), F) \simeq H_c^q((X, S), F)$. Hence if we set $\dim(S) := \dim(Z)$, with the above notations, $H_c^q((X, S), F) = 0$ for all $q > 2\dim(S)$.

2.6.3 Finiteness of the ℓ-adic cohomology

In this subsection, we assume again that k is algebraically closed. We fix $\mathcal{A}$ a k-affinoid algebra, $\mathcal{X}$ a separated $\mathcal{A}$-scheme of finite type, S a locally closed semi-algebraic subset of $\mathcal{X}^{an}$, and ℓ a prime number different from the characteristic of $\tilde{k}$.

In this situation, we have seen in proposition 2.3.2 that for $n \geq 0$, the groups $H^q_c(S, \mathbb{Z}/\ell^n\mathbb{Z})$ are finite (we remind that the notation $H^q_c(S, \mathbb{Z}/\ell^n\mathbb{Z})$ is a shorthand for $H^q_c((X, S), \underline{\mathbb{Z}/\ell^n\mathbb{Z}})$).

We then set

$$H^q_c(S, \mathbb{Z}_\ell) = \varprojlim_{n>0} H^q_c(S, \mathbb{Z}/\ell^n\mathbb{Z}) \tag{2.15}$$

and

$$H^q_c(S, \mathbb{Q}_\ell) = H^q_c(S, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \tag{2.16}$$

It is a classical fact that proposition 2.3.2 implies that the groups $H^q_c(S, \mathbb{Z}_\ell)$ are finitely generated $\mathbb{Z}_\ell$-modules, and as a consequence, that $H^q_c(S, \mathbb{Q}_\ell)$ are finite-dimensional $\mathbb{Q}_\ell$-vector spaces. For completeness, we give here a proof as simple as possible.

One of the main tool to prove it is the so called **Mittag-Leffler condition**, that we will use in the following form:

Proposition 2.6.4. *Let*

$$0 \to (A_n)_{n\geq 0} \to (B_n)_{n\geq 0} \to (C_n)_{n\geq 0} \to 0$$

be an exact sequence of inverse systems of finite abelian groups indexed by $n \in \mathbb{N}$. This means that for each $n \in \mathbb{N}$, the sequence $0 \to A_n \to B_n \to C_n \to 0$ is exact. Then the sequence

$$0 \to \varprojlim_{n\geq 0} A_n \to \varprojlim_{n\geq 0} B_n \to \varprojlim_{n\geq 0} C_n \to 0$$

is exact.

An easy proof can be find as an exercise in [Liu06, exercise 1.3.15], and more general statements can be found in [Gro61, Chapter 0 §13.1]. The finiteness condition is necessary. For instance if K is a field, the morphism of inverse systems:

$$(K[X])_{n\geq 0} \to (K[X]/(X^n))_{n\geq 0}$$

is surjective (the left-hand side is the constant inverse system $K[X]$). However, after taking the inverse limit we obtain the morphism

$$K[X] \to K[|X|]$$

which is not surjective.

Definition 2.6.5. A $\mathbb{Z}_\ell$-module M is called complete and separated (with respect to the ℓ-adic topology) if the canonical map

$$\pi : M \to \widehat{M} := \varprojlim_{k\geq 1} M/\ell^k M$$

is an isomorphism.

Proposition 2.6.6. *Let us consider a projective system of abelian groups*

$$M_1 \xleftarrow{d_1} M_2 \leftarrow \cdots \xleftarrow{d_{n-1}} M_n \xleftarrow{d_n} \cdots$$

where each M_n is a finite $\mathbb{Z}/\ell^n\mathbb{Z}$-module. Then

$$M := \varprojlim_{n\geq 1} M_n$$

is a complete and separated $\mathbb{Z}_\ell$-module.

Proof. We must show that

$$\pi : M \to \widehat{M} = \varprojlim_{k\geq 1} M/\ell^k M$$

is an isomorphism.

We first prove that π is injective. If $x = (x_n) \in M$, and $\pi(x) = 0$, this means that $x \in \ell^k M$ for all k. Since for each n, M_n is a $\mathbb{Z}/\ell^n\mathbb{Z}$-module, taking $k = n$, this implies that $x_n \in \ell^n M_n$, so $x_n = 0$ for all n, hence $x = 0$.

Let us now prove that π is surjective. For this, we consider a Cauchy sequence $(y^{(k)})_{k\geq 1}$ in M, such that for all $j \geq k$, $y^{(j)} \equiv y^{(k)} \mod \ell^k M$. In particular, if $j \geq k$, this implies that for all n,

$$y_n^{(j)} \equiv y_n^{(k)} \mod \ell^k M_n. \tag{2.17}$$

Now all we have to do is to find some $x \in M$ such that for all $k \geq 1$,

$$x \equiv y^{(k)} \mod \ell^k M.$$

First, we define $x = (x_n)$ by

$$x_n = (y_n^{(n)})_{n\geq 1}.$$

Thus we obtain:

$$d_n(x_{n+1}) = d_n(y_{n+1}^{(n+1)}) = y_n^{(n+1)} \equiv y_n^{(n)} \mod \ell^n M_n,$$

the last congruence being a consequence of (2.17). But since M_n is a $\mathbb{Z}/\ell^n\mathbb{Z}$-module, $l^n M_n = \{0\}$, thus $d_n(x_{n+1}) = y_n^{(n)} = x_n$. Hence $(x_n) \in M$.

It is now sufficient to show that $x \equiv y^{(k)} \mod \ell^k M$ for all $k \geq 1$. For this, let us consider some $n \in \mathbb{N}^*$. Then

$$x_n - y_n^{(k)} = y_n^{(n)} - y_n^{(k)}.$$

If $n < k$, then according to (2.17), $y_n^{(n)} - y_n^{(k)} \in \ell^n M_n$ and since $\ell^n M_n = \{0\}$, $y_n^{(n)} = y_n^{(k)}$, so in particular $y_n^{(n)} \equiv y_n^{(k)} \mod \ell^k M_k$. If $n \geq k$, still according to (2.17), $y_n^{(n)} - y_n^{(k)} \in \ell^k M_k$. So in any case $(x - y^{(k)})_n \in \ell^k M_n$.

But since the groups M_n are all finite, according to the Mittag Leffler condition, we have an isomorphism

$$\begin{array}{rccc} \alpha : & \ell^k(\varprojlim_{n\geq 1} M_n) & \xrightarrow{\sim} & \varprojlim_{n\geq 1}(\ell^k M_n) \\ & \ell^k(x_n)_{n\geq 0} & \mapsto & (\ell^k x_n)_{n\geq 0}. \end{array}$$

Indeed α is by definition injective. Moreover, let us consider the map

$$\begin{array}{rccc} \beta : & \varprojlim_{n\geq 1}(M_n) & \to & \varprojlim_{n\geq 1}(\ell^k M_n) \\ & (x_n)_{n\geq 0} & \mapsto & (\ell^k x_n)_{n\geq 0}. \end{array}$$

Since β is induced by a surjective map of inverse systems, according to the Mittag-Leffler condition, β is surjective. By definition, for $x \in \varprojlim_{n\geq 1}(M_n)$, $\beta(x) = \alpha(\ell^k x)$, so α is also surjective.

Hence $x - y^{(k)} \in \ell^k M$ which concludes the proof. $\square$

Lemma 2.6.7. *Let M be a complete and separated $\mathbb{Z}_\ell$-module. Then M is finitely generated if and only if $M/\ell M$ is finite.*

Proof. First, if M is a finitely generated $\mathbb{Z}_\ell$-module, $M/\ell M$ is a finitely generated $\mathbb{Z}/\ell\mathbb{Z}$-module, hence is finite.

Conversely, if $M/\ell M$ is generated by some elements $m_1, \dots, m_N$ from M, we show by induction on n that for each $n \geq 0$, $\ell^n M/(\ell^{n+1}M)$ is generated by $\ell^n(m_1, \dots m_N)$. Indeed, this is true by hypothesis for $n = 0$. Now, if $n > 0$, and $x \in \ell^n M$, say $x = \sum_{i=1}^{N} \ell^n x_i$, then $x = \ell \sum_{i=1}^{N} \ell^{n-1} x_i$ and by induction hypothesis, there exists $y \in \ell^{n-1}(m_1 \dots m_N)$ such that $\sum_{i=1}^{N} \ell^{n-1} x_i \equiv y \mod \ell^n M$. Hence $\ell y \in \ell^n(m_1 \dots m_N)$ and $\ell y \equiv \sum_{i=1}^{N} \ell^n x_i \mod \ell^{n+1} M$.

Hence if $x \in M$, one can inductively define a sequence $(x_n)_{n \geq 0}$ such that $x_n \in (m_1 \dots m_N)$, $x_n = x \mod \ell^n M$ and $x_{n+1} - x_n \in \ell^n(m_1 \dots m_n)$. Hence in $\mathbb{Z}_\ell(m_1 \dots m_N)$, (x_n) has a limit which is x.

□

Proposition 2.6.8. *The groups $H^q_c(S, \mathbb{Z}_\ell)$ are finitely generated $\mathbb{Z}_\ell$-modules. Hence, $H^q_c(S, \mathbb{Q}_\ell)$ is a finitely generated vector space for all q, and $H^q_c(S, \mathbb{Q}_\ell) = \{0\}$ for $q > 2d$, where d is the dimension of $\mathcal{X}^{an}$.*

Proof. According to proposition 2.3.2 and 2.6.6, $H^q_c(S, \mathbb{Z}_\ell)$ is a complete $\mathbb{Z}_\ell$-module. So according to lemma 2.6.7, it only remains to prove that $H^q_c(S, \mathbb{Z}_\ell)/\ell H^q_c(S, \mathbb{Z}_\ell)$ is finite. Let us prove this.

For each $n \geq 0$ we have the exact sequence of groups

$$0 \to \mathbb{Z}/\ell^n\mathbb{Z} \xrightarrow{\mu_n} \mathbb{Z}/\ell^{n+1}\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/\ell\mathbb{Z} \to 0$$

where

$$\begin{array}{cccc} \mu_n : & \mathbb{Z}/\ell^n\mathbb{Z} & \to & \mathbb{Z}/\ell^{n+1}\mathbb{Z} \\ & x \bmod \ell^n & \mapsto & \ell x \bmod \ell^{n+1} \end{array}$$

and π is the reduction map. If we take the long exact sequence in cohomology associated to this, we get the long exact sequence of projective systems:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \vdots \\ \cdots \to & & H^i_c(S, \mathbb{Z}/\ell^n\mathbb{Z}) & \xrightarrow{\mu_n} & H^i_c(S, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) & \longrightarrow & H^i_c(S, \mathbb{Z}/\ell\mathbb{Z}) & \longrightarrow & H^{i+1}_c(S, \mathbb{Z}/\ell^n\mathbb{Z}) \cdots \to \\ & & \alpha \uparrow & & \beta \uparrow & & \| & & \uparrow \\ \cdots \to & & H^i_c(S, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) & \xrightarrow{\mu_{n+1}} & H^i_c(S, \mathbb{Z}/\ell^{n+2}\mathbb{Z}) & \longrightarrow & H^i_c(S, \mathbb{Z}/\ell\mathbb{Z}) & \longrightarrow & H^{i+1}_c(S, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \cdots \to \\ & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

where the first two vertical arrows α and β correspond to the natural projections. In addition, one checks that the composite $\mu_n \circ \alpha$ is just multiplication by ℓ. By the previous section, the groups are all finite, so the functor $\varprojlim_{n \geq 0}$ is exact (this is a particular case of the Mittag-Leffler condition). So, once we apply the functor $\varprojlim_{n \geq 0}$, we obtain the long exact sequence:

$$\cdots \to H^i_c(S, \mathbb{Z}_\ell) \xrightarrow{\times \ell} H^i_c(S, \mathbb{Z}_\ell) \longrightarrow H^i_c(S, \mathbb{Z}/\ell\mathbb{Z}) \longrightarrow H^{i+1}_c(S, \mathbb{Z}_\ell) \cdots \to$$

In particular , this implies that we have an injection:

$$0 \to H^q_c(S, \mathbb{Z}_\ell)/(\ell.H^q_c(S, \mathbb{Z}_\ell)) \to H^q_c(S, \mathbb{Z}/\ell\mathbb{Z})$$

and since $H^q_c(S, \mathbb{Z}/\ell\mathbb{Z})$ is finite, we conclude that $H^q_c(S, \mathbb{Z}_\ell)/(\ell.H^q_c(S, \mathbb{Z}_\ell))$ is finite. □

Remark 2.6.9. As a consequence, the groups $H^q_c(S, \mathbb{Z}_\ell)$ being finitely generated $\mathbb{Z}_\ell$ modules are compact spaces when equipped with the ℓ-adic topology. Moreover, there exists an isomorphism of $\mathbb{Z}_\ell$-modules

$$H^q_c(S, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell^m \times H$$

where $m \in \mathbb{N}$ and H is a finite abelian group. It follows there is a commutative diagram:

$$\begin{array}{ccc} H^q_c(S, \mathbb{Z}_\ell) & \xrightarrow{\sim} & \mathbb{Z}_\ell^m \times H \\ \downarrow & & \downarrow \\ H^q_c(S, \mathbb{Q}_\ell) & \xrightarrow{\sim} & \mathbb{Q}_\ell^m \end{array} \tag{2.18}$$

where the vertical arrows are induced by tensor product. Likewise, being finitely generated $\mathbb{Q}_\ell$ vector spaces, the groups $H^q_c(S, \mathbb{Q}_\ell)$ have a canonical topology, which is inherited by the topology of $H^q_c(S, \mathbb{Z}_\ell)$ after tensor product: the image of $H^q_c(S, \mathbb{Z}_\ell)$ in $H^q_c(S, \mathbb{Z}_\ell)$ is an open and compact subgroup, and all arrows in (2.18) are continuous. It also follows that the map induced by tensor product:

$$\mathrm{End}_{\mathbb{Z}_\ell}(H^q_c(S, \mathbb{Z}_\ell)) \to \mathrm{End}_{\mathbb{Q}_\ell}(H^q_c(S, \mathbb{Q}_\ell))$$

is continuous.

Lemma 2.6.10. *Let $(M_n)_{n\geq 0}$ be an inverse system such that each M_n is a finite $\mathbb{Z}/\ell^n\mathbb{Z}$-module and let $M := \varprojlim_{n\geq 0} M_n$. Let us assume that M is a finitely generated $\mathbb{Z}_\ell$-module. Let $K \in \mathbb{N}$ and let*

$$\mathcal{U} := \ell^K M.$$

Then there exists some $N \in \mathbb{N}$ such that

$$\{(x_n) \in M \mid x_N = 0\} \subset \ell^K M.$$

In other words, on M, the ℓ-adic topology coincides with the linear topology generated by the family of open subgroups $\{(x_n) \in M \mid x_N = 0\}_{N\in\mathbb{N}}$, because conversely $\{(x_n) \in M \mid x_N = 0\} \subset l^N M$.

Proof. Remark that as a finitely generated $\mathbb{Z}_\ell$-module, M equipped with the ℓ-adic topology is a compact space, and $l^K m$ is an open subspace. Then for $N \in \mathbb{N}$, $\{(x_n) \in M \mid x_N = 0\}$ is a closed subset of M, for instance because it is the kernel of the map $M \to M_N$ which is continuous because it is a morphism between finitely generated $\mathbb{Z}_\ell$-modules. Moreover

$$\bigcap_{N\geq 0} \{(x_n) \in M \mid x_N = 0\} = \{0\}$$

by definition.

Hence since $\ell^K M$ is a neighbourhood of $\{0\}$, by compactness of M, there exists some $N \in \mathbb{N}$ such that $\{(x_n) \in M \mid x_N = 0\} \subset \ell^K M$. □

Proposition 2.6.11. *With the above notations, $H^q_c(S, \mathbb{Z}_\ell)$ and $H^q_c(S, \mathbb{Q}_\ell)$ are continuous Galois modules.*

Proof. According to remark 2.6.9, it is enough to prove that $H_c^q(S,\mathbb{Z}_\ell)$ are continuous Galois modules. And in order to prove this, it is enough to prove that for $x \in H_c^q(S,\mathbb{Z}_\ell)$, the map

$$\begin{array}{ccc} \mathrm{Gal}(k^{sep}/k) & \to & H_c^q(S,\mathbb{Z}_\ell) \\ g & \mapsto & g.x \end{array}$$

is continuous. Let then $\mathcal{U}$ be some neighbourhood of $g.x$ in $H_c^q(S,\mathbb{Z}_\ell)$. Then there exists $K \in \mathbb{N}$ such that $g.x + \ell^K H_c^q(S,\mathbb{Z}_\ell) \subset \mathcal{U}$. According to lemma 2.6.10, there exists some $N \in \mathbb{N}$ such that

$$\{(u_n) \in H_c^q(S,\mathbb{Z}_\ell) \mid u_N = 0 \in H_c^q(S,\mathbb{Z}/\ell^N\mathbb{Z})\} \subset \ell^K H_c^q(S,\mathbb{Z}_\ell).$$

Then according to section 2.6.1, $\mathrm{Gal}(k^{sep}/k)$ acts continuously on $H_c^q(S,\mathbb{Z}/\ell^N\mathbb{Z})$. Concretely, there exists a neighbourhood $\mathcal{W}$ of g such that for all $w \in \mathcal{W}$, $w.x_N - g.x_N = 0 \in H_c^q(S,\mathbb{Z}/\ell^N\mathbb{Z})$. So $w.x - g.x \in \ell^K H_c^q(S,\mathbb{Z}_\ell)$, i.e. $w.x \in \mathcal{U}$, which proves that the action of $\mathrm{Gal}(k^{sep}/k)$ is continuous. ☐

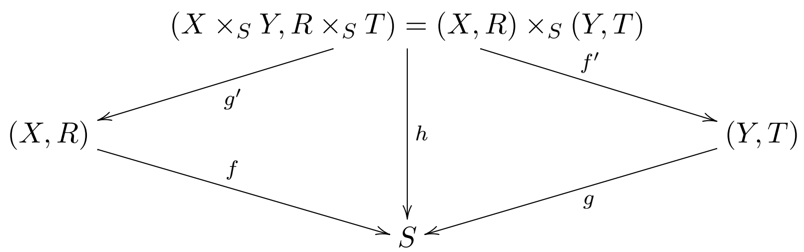
Using again the exactness of $\varprojlim$ when all the groups are finite, and the long exact sequence (2.6), when S is a locally closed semi-algebraic subset, $V \subseteq S$ is a semi-algebraic subset which is open in S and $F = S \setminus V$, then we get a long exact sequence of continuous Galois modules:

$$\cdots \to H_c^i(V,\mathbb{Q}_\ell) \to H_c^i(S,\mathbb{Q}_\ell) \to H_c^i(F,\mathbb{Q}_\ell) \to \cdots$$

2.6.4 Künneth Formula

Definition 2.6.12. Let Λ be a ring. A complex $M^\bullet$ of Λ-modules is called *strictly perfect* if it is bounded, and for all n, M^n is a finitely generated projective Λ-module.

Proposition 2.6.13. *Let*



be a cartesian square of k-germs, where R (resp. T) is locally closed in X (resp. in Y), X, Y and S being some Hausdorff k-analytic spaces. Let $\mathcal{F} \in \mathcal{D}^-(X,\mathbb{Z}/\ell^n\mathbb{Z})$ and $\mathcal{G} \in \mathcal{D}^-(Y,\mathbb{Z}/\ell^n\mathbb{Z})$. Then there is a canonical isomorphism:

$$\mathbf{R}f_!\mathcal{F} \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} \mathbf{R}g_!\mathcal{G} \simeq \mathbf{R}h_!\left((g'^*\mathcal{F}) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} (f'^*(\mathcal{G})\right)$$

Proof. Since R is locally closed, $R = U \cap F$ where U is an open subset of X, and F a closed subset, so R is closed in U, and since the inclusion $(U,R) \to (X,R)$ is a quasi-immersion, replacing X by U, we can assume that R is closed in X. Let us set $U := X \setminus R$ the complementary open subset.

In a first step, let us assume that $T = Y$, that is to say that $(Y,T) = (Y,Y) \simeq Y$. Remark that (X,U) and U are isomorphic as k-$\mathscr{G}$erms. We then consider the following three cartesian diagrams:

$$\begin{array}{ccccc} & & (X,R)\times_S Y & & \\ & \swarrow^{g'} & \downarrow h & \searrow^{f'} & \\ (X,R) & & & & Y \\ & \searrow_{f} & & \swarrow_{g} & \\ & & S & & \end{array}$$

$$\begin{array}{ccccc} & & U\times_S Y & & \\ & \swarrow^{g'_U} & \downarrow h_U & \searrow^{f'_U} & \\ U & & & & Y \\ & \searrow_{f_U} & & \swarrow_{g_U} & \\ & & S & & \end{array}$$

$$\begin{array}{ccccc} & & X\times_S Y & & \\ & \swarrow^{g'_X} & \downarrow h_X & \searrow^{f'_X} & \\ X & & & & Y \\ & \searrow_{f_X} & & \swarrow_{g_X} & \\ & & S & & \end{array}$$

We then obtain a commutative diagram of distinguished triangles:

$$\begin{array}{ccc}
\Big(\mathbf{R}f_{U!}(\mathcal{F}_{|U})\Big) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} (\mathbf{R}g_!\mathcal{G}) & \xrightarrow{1} & \mathbf{R}h_{U!}\left(g'^*_U(\mathcal{F}_{|U}) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} (f'^*_U(\mathcal{G}))\right) \\
\downarrow & & \downarrow \\
(\mathbf{R}f_{X!}(\mathcal{F})) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} (\mathbf{R}g_!\mathcal{G}) & \xrightarrow{2} & \mathbf{R}h_{X!}\left(g'^*_X\mathcal{F} \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} (f'^*_X(\mathcal{G}))\right) \\
\downarrow & & \downarrow \\
\Big(\mathbf{R}f_!(\mathcal{F}_{|(X,R)})\Big) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} (\mathbf{R}g_!\mathcal{G}) & \xrightarrow{3} & \mathbf{R}h_!\left(g'^*(\mathcal{F}_{|(X,R)}) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} (f'^*(\mathcal{G}))\right) \\
\downarrow{\scriptstyle [-1]} & & \downarrow{\scriptstyle [-1]}
\end{array}$$

According to [Ber93, 7.7.3] the arrows 1 and 2 are isomorphisms. So 3 (which is constructed in the same way as in *loc.cit.*) is also an isomorphism.

Next, if (Y,T) is a locally closed k-germ, as above we can assume that T is closed in Y, so that if we set $V := Y \setminus T$, V is an open subset of Y and $V \to (Y,V)$ is a quasi-immersion. Hence according to the first step, the proposition holds for the k-germs (Y,V) and Y, so using again the distinguished triangle associated to (Y,V) and (Y,T), we can conclude. □

Exactly in the same way, we can generalize [Ber93, 5.3.10] to k-germs:

Proposition 2.6.14. *Let $\varphi : Y \to X$ be a Hausdorff morphism of finite dimension, $\mathcal{G} \in D^b(Y, \mathbb{Z}/\ell^n\mathbb{Z})$ of finite* Tor*-dimension, and $\mathcal{F} \in D(X, \mathbb{Z}/\ell^n\mathbb{Z})$. Let T be a locally closed subspace of Y, and let us set $f = \varphi_{|(Y,T)}$. Then $\mathbf{R}f_!(\mathcal{G}_{|(Y,T)})$ is also of finite* Tor*-dimension, and there is a canonical isomorphism*

$$\mathcal{F} \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} \mathbf{R}f_!(\mathcal{G}_{|(Y,T)}) \simeq \mathbf{R}f_!(f^*(\mathcal{F}) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} \mathcal{G}_{|(Y,T)}).$$

We now apply proposition 2.6.13 to the following situation: we assume that $S = \mathcal{M}(k)$ and we consider the constant sheaves

$$F = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}.$$

In that case $\mathbf{R}f_! = \mathbf{R}\Gamma_c$, and we have the following isomorphism in $\mathcal{D}^-(\mathbb{Z}/\ell^n\mathbb{Z} - \mathrm{Mod})$:

$$\mathbf{R}\Gamma_c((X,R), \mathbb{Z}/\ell^n\mathbb{Z}) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} \mathbf{R}\Gamma_c((Y,T), \mathbb{Z}/\ell^n\mathbb{Z}) \simeq \mathbf{R}\Gamma_c((X \times Y, R \times T), \mathbb{Z}/\ell^n\mathbb{Z}). \quad (2.19)$$

Our goal is now to pass from $\mathbb{Z}/\ell^n\mathbb{Z}$ coefficients to $\mathbb{Q}_\ell$ coefficients which is achieved in proposition 2.6.17. The following arguments are a rewriting of the exposition of the ℓ-adic Künneth formula for étale cohomology of schemes made in [Mil80, VI 8].

Using proposition 2.6.14 with $\mathcal{F} = \mathbb{Z}/\ell^{n-1}\mathbb{Z}$ and $\mathcal{G} = \mathbb{Z}/\ell^n\mathbb{Z}$ yields the following isomorphism in $D^-(\mathbb{Z}/\ell^{n-1}\mathbb{Z} - \mathrm{Mod})$:

$$\mathbf{R}\Gamma_c((X,R), \mathbb{Z}/\ell^n\mathbb{Z}) \overset{L}{\underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes}} \mathbb{Z}/\ell^{n-1}\mathbb{Z} \simeq \mathbf{R}\Gamma_c((X,R), \mathbb{Z}/\ell^{n-1}\mathbb{Z}) \quad (2.20)$$

In what follows, we will work with complexes $M^\bullet$ of $\mathbb{Z}_\ell$ (resp. $\mathbb{Z}/\ell^n\mathbb{Z}$) modules. According to the context, we will either see $M^\bullet$ as a complex of modules, or as its image in the derived category $\mathcal{D}(\mathbb{Z}_\ell - \mathrm{Mod})$ (resp. $\mathcal{D}(\mathbb{Z}/\ell^n\mathbb{Z} - \mathrm{Mod})$). For instance when we will consider projective limits $\varprojlim_n M_n^\bullet$, this will always mean that the $M_n^\bullet$'s are complexes of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules. In the same way, if $M^\bullet$ and $N^\bullet$ are complexes of $\mathbb{Z}_\ell$-modules, $M^\bullet \otimes_{\mathbb{Z}_\ell} N^\bullet$ will denote the total tensor product of complexes of $\mathbb{Z}_\ell$-modules, whereas $M^\bullet \overset{L}{\otimes}_{\mathbb{Z}_\ell} N^\bullet$ will denote the total tensor product of $M^\bullet$ and $N^\bullet$ seen as objects of the derived category.

Now we need the following lemma:

Lemma 2.6.15. *For each $n \geq 1$, let $A_n^\bullet$ and $B_n^\bullet$ be strictly perfect complexes of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, and for each $n \geq 2$ let $\varphi_n : A_n^\bullet \to A_{n-1}^\bullet$ (resp. $\psi_n : B_n^\bullet \to B_{n-1}^\bullet$) be a morphism of complex of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, such that the canonical morphism $A_n^\bullet \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} \mathbb{Z}/\ell^{n-1}\mathbb{Z} \to A_{n-1}^\bullet$ (resp. $B_n^\bullet \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} \mathbb{Z}/\ell^{n-1}\mathbb{Z} \to B_{n-1}^\bullet$) is a quasi-isomorphism. Then there is a canonical isomorphism in $\mathcal{D}(\mathbb{Z}_\ell - Mod)$:*

$$\varprojlim_{n \geq 1}(A_n^\bullet \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} B_n^\bullet) \simeq (\varprojlim_{n \geq 1} A_n^\bullet) \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} (\varprojlim_{n \geq 1} B_n^\bullet)$$

Proof. According to [FK88, I 12.5], there exists a strictly perfect complex $A^\bullet$ of $\mathbb{Z}_\ell$-modules and for each n a quasi-isomorphism $\alpha_n : A^\bullet/(\ell^n A^\bullet) \to A_n^\bullet$ such that for all n the following diagram commutes up to homotopy:

$$\begin{array}{ccc} A^\bullet/(\ell^n A^\bullet) & \xrightarrow{\alpha_n} & A_n^\bullet \\ \downarrow{\scriptstyle red} & & \downarrow{\scriptstyle \varphi_n} \\ A^\bullet/(\ell^{n-1} A^\bullet) & \xrightarrow{\alpha_{n-1}} & A_{n-1}^\bullet \end{array} \quad (2.21)$$

and likewise there exists a strictly perfect complex of $\mathbb{Z}_\ell$-modules $B^\bullet$ and some quasi-isomorphisms $\beta_n : B^\bullet/(\ell^n B^\bullet) \to B^\bullet_n$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} B^\bullet/(\ell^n B^\bullet) & \xrightarrow{\beta_n} & B^\bullet_n \\ \downarrow{\scriptstyle red} & & \downarrow{\scriptstyle \psi_n} \\ B^\bullet/(\ell^{n-1} B^\bullet) & \xrightarrow{\beta_{n-1}} & B^\bullet_{n-1} \end{array} \tag{2.22}$$

Remind that if M is a $\mathbb{Z}_\ell$-module of finite type, there is a functorial isomorphism:

$$M \xrightarrow{\sim} \varprojlim_{n\geq 1} M/(\ell^n M) \tag{2.23}$$

We then obtain the following quasi-isomorphisms:

$$A^\bullet \xrightarrow{\sim} \varprojlim_n (A^\bullet/(\ell^n A^\bullet)) \xrightarrow{\sim} \varprojlim_n (A^\bullet_n), \tag{2.24}$$

where the first arrow is an isomorphism of complexes according to (2.23) and the second arrow is a quasi-isomorphism according to Mittag Leffler condition and the fact that all the modules involved are of finite type. The similar results holds for $B^\bullet$.

We then obtain the following sequence of isomorphisms in $\mathcal{D}(\mathbb{Z}_\ell - \text{Mod})$:

$$(\varprojlim_n A^\bullet_n) \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} (\varprojlim_n B^\bullet_n) \xleftarrow{\sim} A^\bullet \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} B^\bullet \tag{2.25}$$

$$\simeq A^\bullet \underset{\mathbb{Z}_\ell}{\otimes} B^\bullet \tag{2.26}$$

$$\xrightarrow{\sim} \varprojlim_n \left((A^\bullet \underset{\mathbb{Z}_\ell}{\otimes} B^\bullet)/(\ell^n (A^\bullet \underset{\mathbb{Z}_\ell}{\otimes} B^\bullet)) \right) \tag{2.27}$$

$$\xrightarrow{\sim} \varprojlim_n \left((A^\bullet/(\ell^n A^\bullet)) \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} (B^\bullet/(\ell^n B^\bullet)) \right) \tag{2.28}$$

$$\xrightarrow{\sim} \varprojlim_n \left(A^\bullet_n \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} (B^\bullet/(\ell^n B^\bullet)) \right) \tag{2.29}$$

$$\xleftarrow{\sim} \varprojlim_n (A^\bullet_n \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} B^\bullet_n). \tag{2.30}$$

The isomorphism (2.25) holds thanks to (2.24), (2.26) holds because $A^\bullet$ and $B^\bullet$ are flat, (2.27) is remark (2.23), (2.28) is base change for tensor product.

Finally to obtain (2.29) we take the tensor product of the first (resp. second) line of diagram (2.21) with $B^\bullet/(\ell^n B^\bullet)$ (resp. $B^\bullet/(\ell^{n-1} B^\bullet)$). The resulting diagram still commutes up to homotopy and since $B^\bullet/(\ell^n B^\bullet)$ is a projective complex, the horizontal lines are still quasi-isomorphisms. Hence (thanks to Mittag Leffler condition), we obtain (2.29).

Similarly, for (2.30), we take the tensor product of the first (resp. second) line of diagram (2.22) with $A^\bullet_n$ (resp. $A^\bullet_{n-1}$). Since $A^\bullet_n$ is a projective complex, the horizontal lines remain quasi-isomorphisms and we can conclude with the same argument. □

Remark 2.6.16. Note that we've implicitly used the following result: if $M_1^\bullet$, $M_2^\bullet$ and $M_3^\bullet$ are bounded above complexes of Λ-module such that $M_3^\bullet$ is projective, and $f : M_1^\bullet \to M_2^\bullet$ is a quasi-isomorphism, then $f \otimes id : M_1^\bullet \otimes M_3^\bullet \to M_2^\bullet \otimes M_3^\bullet$ is a quasi-isomorphism [Wei94, 10.6.2]

Proposition 2.6.17. *Let (X,R), (Y,T) be k-germs such that for all n, $\mathbf{R}\Gamma_c((X,R),\mathbb{Z}/\ell^n\mathbb{Z})$ and $\mathbf{R}\Gamma_c((Y,T),\mathbb{Z}/\ell^n\mathbb{Z})$ have finite cohomology groups. Then the cohomology groups of $\mathbf{R}\Gamma_c((X,R)\times(Y,T),\mathbb{Z}/\ell^n\mathbb{Z})$ are also finite and for all $r\geq 0$ we have a canonical isomorphism:*

$$H_c^r\left((X\times Y,R\times T),\mathbb{Q}_\ell\right)\simeq\bigoplus_{p+q=r}H_c^p\left((X,R),\mathbb{Q}_\ell\right)\otimes H_c^q\left((Y,T),\mathbb{Q}_\ell\right).$$

Proof. The complexes $\mathbf{R}\Gamma_c((X,R),\mathbb{Z}/\ell^n\mathbb{Z})$ and $\mathbf{R}\Gamma_c((Y,T),\mathbb{Z}/\ell^n\mathbb{Z})$ have bounded cohomology groups, are of finite type by hypothesis, and according to proposition 2.6.14 are of finite Tor-dimension, so we can choose some resolutions by some strictly perfect complexes of the projective systems:

$$\begin{array}{ccc} K_n^\bullet & \to & \mathbf{R}\Gamma_c((X,R),\mathbb{Z}/\ell^n\mathbb{Z}) \\ P_n^\bullet & \to & \mathbf{R}\Gamma_c((Y,T),\mathbb{Z}/\ell^n\mathbb{Z}) \\ Q_n^\bullet & \to & \mathbf{R}\Gamma_c((X,R)\times(Y,T),\mathbb{Z}/\ell^n\mathbb{Z}). \end{array}$$

In addition, according to (2.19) we can find up to homotopy a quasi-isomorphism, of projective systems:

$$K_n^\bullet \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} P_n^\bullet \simeq Q_n^\bullet. \tag{2.31}$$

Moreover according to (2.20), $K_n^\bullet$ and $P_n^\bullet$ fulfil the hypothesis of lemma 2.6.15. We then denote by

$$K^\bullet=\varprojlim_n(K_n^\bullet)$$
$$P^\bullet=\varprojlim_n(P_n^\bullet)$$
$$Q^\bullet=\varprojlim_n(Q_n^\bullet).$$

Remark that (thanks to Mittag Leffler property again)

$$H^p(K^\bullet)\simeq H_c^p((X,R),\mathbb{Z}_\ell) \tag{2.32}$$
$$H^p(P^\bullet)\simeq H_c^p((Y,T),\mathbb{Z}_\ell) \tag{2.33}$$
$$H^p(Q^\bullet)\simeq H_c^p((X\times Y,R\times T),\mathbb{Z}_\ell). \tag{2.34}$$

In $\mathcal{D}(\mathbb{Z}_\ell-\mathrm{Mod})$ we consider the following sequence of isomorphisms:

$$K^\bullet \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} P^\bullet \simeq \varprojlim_n(K_n^\bullet) \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} \varprojlim_n(P_n^\bullet) \tag{2.35}$$
$$\simeq \varprojlim_n(K_n^\bullet \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} P_n^\bullet) \tag{2.36}$$
$$\simeq \varprojlim_n(Q_n^\bullet) \tag{2.37}$$
$$\simeq Q^\bullet. \tag{2.38}$$

The isomorphism (2.35) holds by definition of $K^\bullet$ and $P^\bullet$, (2.36) holds thanks to lemma 2.6.15, (2.37) is just a consequence of (2.31), and (2.38) holds by definition of $Q^\bullet$.

We then obtain the following isomorphisms in $\mathcal{D}(\mathbb{Q}_\ell-\mathrm{Mod})$:

$$\left(K^\bullet \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} \mathbb{Q}_\ell\right) \overset{L}{\underset{\mathbb{Q}_\ell}{\otimes}} \left(P^\bullet \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} \mathbb{Q}_\ell\right) \simeq \left(K^\bullet \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} P^\bullet\right) \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} \mathbb{Q}_\ell \simeq Q^\bullet \overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}} \mathbb{Q}_\ell$$

But since $\mathbb{Q}_\ell$ is flat over $\mathbb{Z}_\ell$, we can replace all the $\overset{L}{\underset{\mathbb{Z}_\ell}{\otimes}}\mathbb{Q}_\ell$ by some $\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$. Finally, since $\mathbb{Q}_\ell$ is a field,

$$H^r\Big((K^\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} (P^\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)\Big) \simeq \bigoplus_{p+q=r} H^p(K^\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^q(P^\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

The result then follows from the isomorphisms (2.32)–(2.34). □

We must mention that the proposition 2.6.13 is functorial in S. So let $(\mathcal{X}, R)$, $(\mathcal{Y}, T)$ be k-germs, with $\mathcal{X}$ (resp. $\mathcal{Y}$) a separated $\mathcal{A}$-scheme (resp. $\mathcal{B}$-scheme) of finite type, $\mathcal{A}$ (resp. $\mathcal{B}$) being some k-affinoid algebra and R (resp. T) being some locally closed semi-algebraic subset of $\mathcal{X}^{an}$ (resp. $\mathcal{Y}^{an}$). Then, there are isomorphisms of Galois modules:

$$\bigoplus_{p+q=r} H_c^p\left((\overline{\mathcal{X}^{an}}, \overline{R}), \mathbb{Q}_\ell\right) \otimes H_c^q\left((\overline{\mathcal{Y}^{an}}, \overline{T}), \mathbb{Q}_\ell\right) \simeq H_c^r\left((\overline{(\mathcal{X}\times\mathcal{Y})^{an}}, \overline{R\times T}), \mathbb{Q}_\ell\right).$$

2.6.5 Statement of the main theorem

We sum up all results of this section:

Theorem 2.6.18. *Let k be a non-Archimedean complete valued field, $\mathcal{A}$ a k-affinoid algebra, $\mathcal{X}$ a separated $\mathcal{A}$-scheme of finite type of dimension d, U a locally closed semi-algebraic subset of $\mathcal{X}^{an}$, and $\ell \neq char(\tilde{k})$ be a prime number. We denote by $\pi : \overline{\mathcal{X}^{an}} \to \mathcal{X}^{an}$ the morphism defined in 2.6.1 and we set $\overline{U} = \pi^{-1}(U)$.*

1. *The groups $H_c^i(\overline{U}, \mathbb{Q}_\ell)$ are finite dimensional $\mathbb{Q}_\ell$-vector spaces, endowed with a continuous Gal(k^{sep}/k)-action, and $H_c^i(\overline{U}, \mathbb{Q}_\ell) = 0$ for $i > 2d$.*
2. *Let $V \subset U$ be a semi-algebraic subset which is open in U, and let $F = U \setminus V$. Then there is a Gal(k^{sep}/k)-equivariant long exact sequence*

$$\cdots \to H_c^i(\overline{V}, \mathbb{Q}_\ell) \longrightarrow H_c^i(\overline{U}, \mathbb{Q}_\ell) \longrightarrow H_c^i(\overline{F}, \mathbb{Q}_\ell) \longrightarrow H_c^{i+1}(\overline{V}, \mathbb{Q}_\ell) \cdots \to$$

3. *For all integer n there are canonical Gal(k^{sep}/k)-equivariant isomorphisms:*

$$\bigoplus_{i+j=n} H_c^i\left(\overline{U}, \mathbb{Q}_\ell\right) \otimes_{\mathbb{Q}_\ell} H_c^j\left(\overline{V}, \mathbb{Q}_\ell\right) \simeq H_c^n\left(\overline{U\times V}, \mathbb{Q}_\ell\right)$$

Chapter 3

Dimension of subanalytic sets

In section 3.1 we make a systematic study of the dimension of subanalytic sets $S \subset \left((K^{\mathrm{alg}})^\circ\right)^n$. Using Berkovich spaces, we prove some results which were already known in [LR00a], but we also give new results such as invariance under subanalytic isomorphism and a formula 3.1.20 which calculates the dimension of a subanalytic set in terms of the dimension of some fibers S_x.

In section 3.2, we just sum up the results of [vdD89] about a similar dimension theory for definable sets in ACVF. We have thought it was interesting to gather these results which nonetheless seem to be well known by the experts.

3.1 Dimension of subanalytic sets

We fix a complete non-Archimedean field K. We will denote by $\mathbb{B}^n_K$ the Berkovich closed unit disc $\mathcal{M}(K\langle X_1, \dots, X_n\rangle)$.

In this part, we expose and extend the theory of dimension of subanalytic sets of $\left((K^{\mathrm{alg}})^\circ\right)^n$ as introduced in [LR00a]. If $S \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ is a subanalytic set, its dimension $\dim(S)$ is defined [LR00a, 2.1] as the greatest integer d such that there exists a coordinate projection $\pi : \left((K^{\mathrm{alg}})^\circ\right)^n \to \left((K^{\mathrm{alg}})^\circ\right)^d$ for which $\pi(S)$ has nonempty interior.

Many of the results we present were already known, however some were not (at least not to our knowledge). First, we explain how the dimension of a subanalytic set $S \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ has a nice interpretation in terms of S_{Berko}, a subset of $\mathbb{B}^n_K$ that we attach to S. This new characterization enables us to prove some general facts about the dimension of subanalytic sets. Some were already proven in [LR00a], some were not.

Y. Firat Celikler has also published some results about dimension of D-semianalytic sets [Çel05, Çel10]. His results apply to a broader class of sets, namely subsets $X \subset (K^\circ)^n$ where K is not assumed to be algebraically closed (and not even a rank one valuation in [Çel10]). Whereas our major tool is the use of Abhyankar points, Celikler never mentions Berkovich spaces, and uses extensively generalized ring of fractions and a very interesting (for itself) normalization lemma [Çel05, section 5] and [Çel10, section 3]. These articles contain results about dimension of subsets of $(K^\circ)^n$ in terms of the dimension of some fibers [Çel05, theorem 6.6] and [Çel10, theorem 5.1], which happen to be quite particular cases of our proposition 3.1.20 (but with more general hypothesis).

Let us finally mention that the first order theory defined by the analytic language $\mathcal{L}^D_{an}$ is a C-minimal theory [LR98]. C-minimal theories have been studied in [HM94], and some results of the dimension theory of subanalytic sets follow from the general treatment which is made in [HM94].

3.1.1 Subsets of $\left((K^{\mathbf{alg}})^\circ\right)^n$ and subsets of $\mathbb{B}_K^n$

In this section K is fixed, and for us an $\mathcal{L}_{an}^D$-formula is a formula built with D_0, D_1 and functions of $S_{m,n}(E,K)$ for some choice of E. Somehow, we should then talk about $\mathcal{L}_{an}^D$-formulas defined over K. Clearly, if $K \to L$ is a non-Archimedean extension, and φ is an $\mathcal{L}_{an}^D$-formula defined over K, then φ is also an $\mathcal{L}_{an}^D$-formula defined over L. We will repeatedly use the following remark.

Remark 3.1.1. Let $K \hookrightarrow L$ be a non-Archimedean extension. Following the discussion we have had in 0.1.2 about characters, there is a continuous map

$$p : L^n \to (\mathbb{A}_K^n)^{\mathrm{an}}. \tag{3.1}$$

If $x = (x_1, \dots, x_n) \in L^n$, then for $f \in K[X]$, $|f(p(x))| = |f(x)|_L$.

Definition 3.1.2. Let φ be an $\mathcal{L}_{an}^D$-formula in the variables $X_1, \dots, X_n$. Let $K \to L$ be a non-Archimedean extension. We set

$$\varphi(L^{\mathrm{alg}}) = \{x \in ((L^{\mathrm{alg}})^\circ)^n \mid (L^{\mathrm{alg}})^\circ \models \varphi(x_1, \dots, x_n)\}.$$

This is by definition a subanalytic set of $((L^{\mathrm{alg}})^\circ)^n$.

By definition, for each subanalytic set $S \subset \left((K^{\mathrm{alg}})^\circ\right)^n$, there exists φ an $\mathcal{L}_{an}^D$-formula in the variables $X_1, \dots, X_n$ such that $S = \varphi(K^{\mathrm{alg}})$.

Definition 3.1.3. If $X \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ is a subanalytic set which is defined by the $\mathcal{L}_{an}^D$-formula φ, i.e. $X = \varphi(L^{\mathrm{alg}})$, we set

$$X(L^{\mathrm{alg}}) := \varphi(L^{\mathrm{alg}}).$$

This is independent of the formula φ thanks to the uniform quantifier elimination theorem.

Definition 3.1.4. Let φ be an $\mathcal{L}_{an}^D$-formula in the variables $X_1, \dots, X_n$. Let $x \in \mathbb{B}_K^n$. We will say that x satisfies φ if

$$\mathcal{H}(x)^{\mathrm{alg}} \models \varphi(x_1, \dots, x_n)$$

where $(x_1, \dots, x_n) \in \mathcal{H}(x)^n$ are the evaluations of the X_i's in $\mathcal{H}(x)$.

Definition 3.1.5. Let φ be an $\mathcal{L}_{an}^D$-formula in the variables $X_1, \dots, X_n$. We set

$$\varphi_{\mathrm{Berko}} = \{x \in \mathbb{B}_K^n \mid x \text{ satisfies } \varphi\}.$$

More generally, if $K \to L$ is a non-Archimedean extension, we set

$$(\varphi_L)_{\mathrm{Berko}} = \{x \in \mathbb{B}_L^n \mid x \text{ satisfies } \varphi\}.$$

Finally, if $S \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ is a subanalytic set defined by a $\mathcal{L}_{an}^D$ formula φ (that is to say, $S = \varphi(K^{alg})$), we set

$$S_{\mathrm{Berko}} := \{x \in \mathbb{B}_K^n \mid x \text{ satisfies } \varphi\}.$$

The last definition depends indeed only on the subanalytic set $S \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ and not on the choice of the formula φ thanks to the uniformity in the quantifier elimination theorem (see 0.5.9). Indeed, if φ and ψ were $\mathcal{L}_{an}^D$-formula such that $\varphi(K^{\mathrm{alg}}) = \psi(K^{\mathrm{alg}})$, thanks to the uniform quantifier elimination theorem, $\varphi \leftrightarrow \psi$, hence $\varphi_{\mathrm{Berko}} = \psi_{\mathrm{Berko}}$.

We then have the following tautological result:

Lemma 3.1.6. *Let φ be an $\mathcal{L}_{an}^D$ quantifier free formula, $K \to L$ some non-Archimedean extension. We consider $p : L^n \to \mathbb{B}_K^n$ as defined in remark 3.1.1. Then*

$$\varphi(L^{alg}) = p^{-1}(\varphi_{Berko})$$

More generally, if $K \to L \to M$ are some non-Archimedean extensions, and if we consider $p : M^n \to \mathbb{B}_L^n$, then

$$\varphi(M) = p^{-1}((\varphi_L)_{Berko}).$$

Proof. Let $m = (m_1, \dots, m_n) \in M^n$, and let $x = |\cdot|_m := p(m) \in \mathbb{B}_L^n$ the associated point in $\mathbb{B}_L^n$. Then there exists a unique non-Archimedean embedding $\mathcal{H}(x) \to L$ such $x_i = m_i$ where we consider the natural maps $K\langle X_1, \dots, X_n\rangle \to \mathcal{H}(x) \to L$ and x_i (resp. m_i) are the images of X_i. Hence, for $f \in \mathcal{L}_{an}^D$, $|f(x)| = |f(m)|$, and m satisfies φ if and only if x does. □

Definition 3.1.7. We will say that a map $\pi : L^n \to L^d$ is a coordinate projection if there exists a sequence $1 \leq i_1 < \dots < i_d \leq n$ such that π is defined by

$$\pi(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_d}).$$

Definition 3.1.8. Let $X \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ and $Y \subset \left((K^{\mathrm{alg}})^\circ\right)^m$ be subanalytic sets. We say that a map $f : X \to Y$ is a **subanalytic map** if its graph $\mathrm{Graph}(f) := \{(x, y) \in X \times Y \mid y = f(x)\}$ is a subanalytic set of $\left((K^{\mathrm{alg}})^\circ\right)^{m+n}$.

We want to list some elementary remarks and compatibility results about all these definitions.

1. We fix $X \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ and $Y \subset \left((K^{\mathrm{alg}})^\circ\right)^m$ some subanalytic sets and $f : X \to Y$ a subanalytic map. Let $K \to L$ be some non-Archimedean extension. Then we can naturally define a subanalytic map $f_L : X(L^{\mathrm{alg}}) \to Y(L^{\mathrm{alg}})$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X(L^{\mathrm{alg}}) & \xrightarrow[f_L]{} & Y(L^{\mathrm{alg}}) \end{array}$$

Indeed, $\mathrm{Graph}(f)$, the graph of f, is a subanalytic set of $\left((K^{\mathrm{alg}})^\circ\right)^{m+n}$ which satisfies the conditions

$$\mathrm{Graph}(f) \subset X \times Y$$
$$\forall x \in X, \exists! y \in Y, (x, y) \in \mathrm{Graph}(f)$$

These two conditions are equivalent to say that $\mathrm{Graph}(f)$ is the graph of function $X \to Y$. Thanks to the uniform quantifier elimination theorem, these two conditions are still satisfied when we pass to L^{alg} :

$$\mathrm{Graph}(f)(L^{\mathrm{alg}}) \subset X(L^{\mathrm{alg}}) \times Y(L^{\mathrm{alg}})$$
$$\forall x \in X(L^{\mathrm{alg}}), \exists! y \in Y(L^{\mathrm{alg}}), (x, y) \in \mathrm{Graph}(f)(L^{\mathrm{alg}})$$

Hence $\mathrm{Graph}(f)(L^{\mathrm{alg}})$ is a subanalytic set of $\left((K^{\mathrm{alg}})^\circ\right)^{m+n}$ which defines a subanalytic map $f_L : X(L^{\mathrm{alg}}) \to Y(L^{\mathrm{alg}})$.

Using exactly the same argument, we can prove that f is injective (resp. surjective, bijective) if and only if f_L is injective (resp. surjective, bijective). Also, with the same ideas, f is continuous if and only if f_L is continuous.

2. Let now $S \subset \left((K^{\mathrm{alg}})^\circ\right)^{m+n}$ and $T \subset \left((K^{\mathrm{alg}})^\circ\right)^{m}$ be some subanalytic sets and let us assume that if $(x_1, \dots, x_{m+n}) \in S$, then $(x_1, \dots, x_m) \in T$. We then introduce the map $\pi : S \to T$ which is the projection on the first m-coordinates. It is then clear that we obtain an associated map

$$S_{\mathrm{Berko}} \xrightarrow{\pi_{\mathrm{Berko}}} T_{\mathrm{Berko}}$$

which is actually induced by the map $\mathbb{B}_K^{m+n} \to \mathbb{B}_K^m$ corresponding to the projection on the first m coordinates. We want to prove that if π is surjective (resp. injective) then π_{Berko} is surjective (resp. injective).

If π is surjective then π_{Berko} is also surjective. To see it, take $y \in T_{\mathrm{Berko}}$. Let then $K \to L$ some non-Archimedean and algebraically closed extension and $(y_1, \dots, y_m) \in T(L)$ such that $y = p(y_1, \dots, y_m)$ where $p : L^n \to \mathbb{B}_K^m$ is the map introduced in 3.1.1. We then obtain the commutative diagram

$$\begin{array}{ccc} S(L) & \longrightarrow & S_{\mathrm{Berko}} \\ {\scriptstyle \pi_L}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{Berko}}} \\ T(L) & \longrightarrow & T_{\mathrm{Berko}} \end{array}$$

Since π is surjective, π_L is also surjective, hence we can find $(x_1, \dots, x_{m+n})$ some antecedent of y by π_L. Then if x is the image of $(x_1, \dots, x_{m+n})$ in S_{Berko}, by assumption $\pi_{\mathrm{Berko}}(x) = y$.

Let us now assume that π is injective. Let then $y \in T_{\mathrm{Berko}}$ and $x, x' \in S_{\mathrm{Berko}}$ such that $\pi_{\mathrm{Berko}}(x) = \pi_{\mathrm{Berko}}(x') = y$. We obtain the diagram of non-Archimedean fields:

$$\begin{array}{ccccc} & & \mathcal{H}(x) & & \\ & \swarrow & & \searrow & \\ \mathcal{H}(x) & & & & \mathcal{H}(x') \end{array}$$

We can then find a non-Archimedean field L which completes this diagram

$$\begin{array}{ccccc} & & \mathcal{H}(x) & & \\ & \swarrow & & \searrow & \\ \mathcal{H}(x) & & & & \mathcal{H}(x') \\ & \searrow & & \swarrow & \\ & & L & & \end{array}$$

Let us then write $(y_1, \dots, y_m) \in \mathcal{H}(y)^m$ the evaluations of the $X_1, \dots, X_m$ in $\mathcal{H}(y)$, and $(x_1, \dots, x_{m+n}) \in \mathcal{H}(x)$ the evaluations of $X_1, \dots, X_{m+n}$ in $\mathcal{H}(x)$ and similarly $(x'_1, \dots, x'_{m+n}) \in \mathcal{H}(x')$ the evaluations of $X_1, \dots, X_{m+n}$ in $\mathcal{H}(x')$. By definition of π, if we work in L, we have $y_i = x_i = x'_i$ for $i = 1 \dots m$. Hence $(x_1, \dots, x_{m+n})$ and $(x'_1, \dots, x'_{m+n}) \in S(L^{\mathrm{alg}})$ and by assumption $p_L(x_1, \dots, x_{m+n}) = p_L(x'_1, \dots, x'_{m+n}) = (y_1, \dots, y_m)$. Since p_L is injective, $(x_1, \dots, x_{m+n}) = (x'_1, \dots, x'_{m+n})$, hence since $x \in S_{\mathrm{Berko}}$ corresponds to the image of $(x_1, \dots, x_{m+n})$ by the map $((L^{\mathrm{alg}})^\circ)^{m+n}$ and similarly for x', it follows that $x = x'$.

3. According to the previous point, if we are in the situation 1. with a subanalytic map $f : X \to Y$, then since the projection $\mathrm{Graph}(f) \to X$ is a bijection, according to 2. we deduce that the induced map $\mathrm{Graph}(f)_{\mathrm{Berko}} \to X_{\mathrm{Berko}}$ is bijective. Hence this allows to define a natural map $f_{\mathrm{Berko}} : S_{\mathrm{Berko}} \to T_{\mathrm{Berko}}$.

   According to 2., f_{Berko} is injective (resp. surjective, bijective) if f is is.

Lemma 3.1.9. *Let $X \subset \left((K^{alg})^\circ\right)^n$ and $Y \subset \left((K^{alg})^\circ\right)^m$. Let $f : X \to Y$ be a subanalytic map. Let $x \in X_{Berko}$ and $y := f_{Berko}(x) \in Y_{Berko}$. Then $d(\mathcal{H}(y)/K) \leq d(H(x)/K)$.*

Of course, if f is a morphism of k-affinoid spaces, then this is true because there is a map of non-Archimedean fields $\mathcal{H}(y) \to \mathcal{H}(x)$. But in our context where f is just a subanalytic map, there is no given map $\mathcal{H}(y) \to \mathcal{H}(x)$.

Proof. Let us consider $(x_1, \dots, x_n) \in X(\mathcal{H}(x)^{\mathrm{alg}})$ where for each i, $x_i \in \mathcal{H}(x)$ is the evaluation of the coordinate X_i in $\mathcal{H}(x)$. Since we have a commutative diagram

$$\begin{array}{ccc} X(\mathcal{H}(x)^{\mathrm{alg}}) & \xrightarrow{f} & Y(\mathcal{H}(x)^{\mathrm{alg}}) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ X_{\mathrm{Berko}} & \xrightarrow[f_{\mathrm{Berko}}]{} & Y_{\mathrm{Berko}} \end{array}$$

$f(x_1, \dots, x_n) \in ((\mathcal{H}(x)^{\mathrm{alg}})^\circ)^m$. Since $y = p(f(x_1, \dots, x_n))$, it follows that there is a non-Archimedean extension $\mathcal{H}(y) \to \mathcal{H}(x)^{\mathrm{alg}}$ and the result follows because

$$d(\mathcal{H}(y)/K) \leq d(\mathcal{H}(x)^{\mathrm{alg}}/K) = d(\mathcal{H}(x)/K).$$

□

3.1.2 Proving general properties with Berkovich points

If $K \to L$ is a non-Archimedean extension and $y = (y_1, \dots, y_n) \in L^n$. We will denote by $K(y)$ the field $K(y_1, \dots, y_n)$. By definition, it satisfies $K \subset K(y) \subset L$.

Definition 3.1.10. If $K \to L$ is an extension of non-Archimedean field, the dimension of L over K is

$$d(L/K) = \mathrm{tr\ deg}(\widetilde{L}/\widetilde{K}) + \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} (|L^*|/|K^*|).$$

We will of often use that if $K \to L \to M$ are extension of non-Archimedean fields

$$d(M/K) = d(M/L) + d(L/K). \tag{3.2}$$

We will also use that

$$d(\widehat{L}/K) = d(L/K). \tag{3.3}$$

There is a good dimension theory for k-analytic spaces (see [Ber90, p 34] and [Duc07]). This dimension has a simple interpretation in terms of the definition 3.1.10: if X is a K-analytic space [Ber93, 2.5.2]:

$$\dim(X) = \sup_{x \in X} d(\mathcal{H}(x)/K). \tag{3.4}$$

We will show below that a similar result also holds for subanalytic sets (3.1.14 (5)).

Definition 3.1.11. If $c \in K^n$, and $(r_1, \dots, r_n) \in \mathbb{R}_+^n$, we define $\eta_{c,(r_1,\dots,r_n)} \in \mathbb{A}_K^{n,an}$ by the formula

$$\begin{array}{cccc} \eta_{c,(r_1,\dots,r_n)}: & K[X] & \to & \mathbb{R}_+ \\ & f = \sum_{\nu=(\nu_1\dots\nu_n)\in\mathbb{N}^n} a_\nu \prod_{i=1}^n (X_i - c_i)^{\nu_i} & \mapsto & |f(\eta_{c,(r_1,\dots,r_n)})| = \max_{\nu\in\mathbb{N}^n} |a_\nu| r^\nu \end{array}$$

where $r^\nu = r_1^{\nu_1} \cdots r_n^{\nu_n}$.

Lemma 3.1.12. *Let $K \hookrightarrow L$ be a non-Archimedean extension. Let $x_1, \dots, x_s \in L^*$ such that the real numbers $r_i := |x_i|$, are $\mathbb{Q}$-linearly independent in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}_+^*/|K^*|$. Let $(y_1, \dots, y_t) \in L^\circ$ such that the $|y_i| = 1$ and $\widetilde{y_i}$ are algebraically independent over $\widetilde{K}$. Then the the image of $(x_1, \dots, x_s, y_1, \dots, y_t) \in L^{s+t}$ in $\mathbb{A}_K^{s+t,an}$ through p is $\eta_{0,(r_1,\dots,r_s,1,\dots,1)}$.*

Proof. We set $K[X, Y] := K[X_1, \dots, X_s, Y_1, \dots, Y_t]$.

Step 1. We claim that if

$$g = \sum_{\mu\in\mathbb{N}^t} g_\mu Y^\mu \in K[Y],$$

then

$$|g(y)| = \|g\|,$$

where we remind that $\|g\| = \max_{\mu\in\mathbb{N}^t} |g_\mu|$. To prove this we can assume that $\|g\| = 1$, i.e. that $g \in K^\circ[Y] \setminus K^{\circ\circ}[Y]$. We can then consider $\tilde{g} = \sum_\mu \tilde{g_\mu} Y^\mu \in \widetilde{K}[Y]$ which is non zero. Now $g(y) \in L^\circ$ and in $\tilde{L}$,

$$\widetilde{g(y)} = \sum_{\mu\in\mathbb{N}^t} \widetilde{g_\mu} \widetilde{y^\mu}.$$

Since by assumption the $\tilde{y_i}$'s are algebraically independent over $\widetilde{K}$, the right hand side must be non zero, so $\widetilde{g(y)} \neq 0$. So $|g(y)| = 1 = \|g\|$.

Step 2. We now consider some element

$$f = \sum_{\nu\in\mathbb{N}^s} X^\nu f_\nu(Y) \in K[X, Y],$$

where each $f_\nu \in K[Y]$. According to step 1, $|x^\nu f_\nu(y)| = r^\nu \|f_\nu\|$, then for $\nu \neq \mu$, with $f_\nu \neq 0$, $|x^\nu f_\nu(y)| \neq |x^\mu f_\mu(y)|$. It then follows from the triangle ultrametric inequality that

$$|f(x, y)| = \max_{\nu\in\mathbb{N}^s} r^\nu \|f_\nu\|$$

which is precisely $|f(\eta_{0,(r_1,\dots,r_s,1,\dots,1)})|$. □

Proposition 3.1.13. *Let $K \hookrightarrow L$ be a non-Archimedean extension, and $y = (y_1, \dots, y_n) \in L^n$. Let $p : L^n \to (\mathbb{A}_K^n)^{an}$ as in remark 3.1.1 and $x := p(y)$. Then if $d(\mathcal{H}(x)/K) = n$, there exists a neighbourhood V of y in L^n such that $p(V) = \{x\}$.*

Such points x of $(\mathbb{A}_K^n)^{\text{an}}$ are sometimes called **Abhyankar points**. We refer to [Poi11, 4.1] for another appearance of these points.

Proof. First note that $d(K(y)/K) = d(\mathcal{H}(x)/K)$ because $\mathcal{H}(y) \simeq \widehat{K(y)}$. Let $s := \text{tr.deg.}(\widetilde{K(y)}/\widetilde{K})$ and $t := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} |K(y)^*|/|K^*|)$.

Then $n = s+t$. Hence for each $i = 1\dots n$ one can introduce some fractions $F_i = \frac{P_i}{Q_i} \in K(X)$ such that for all $i = 1, \dots, n$,

$$Q_i(y) \neq 0.$$

$$\left\{\left|\frac{P_i(y)}{Q_i(y)}\right|\right\}_{i=1\dots t} \text{ are } \mathbb{Q}\text{-linearly independent in } \mathbb{Q}\otimes_{\mathbb{Z}}\left(|K(y)^*|/|K^*|\right).$$

$$\text{For } i = t+1\dots n,\ \left|\frac{P_i(y)}{Q_i(y)}\right| = 1, \text{ and}$$

$$\left\{\left(\widetilde{\frac{P_i(y)}{Q_i(y)}}\right)\right\}_{i=t+1\dots n} \text{ are algebraically independent over } \widetilde{K}.$$

We set $r_i := \left|\frac{P_i(y)}{Q_i(y)}\right|$.

Let now $\mathcal{U}$ be the affine subset of $\mathbb{A}^n_K$ defined by $\mathcal{U} = \{z \in \mathbb{A}^n_K \mid Q_i(z) \neq 0,\ i = 1\dots n\}$. Then

$$F = (F_1, \dots, F_n) : \mathcal{U} \to \mathbb{A}^n_K$$

is a regular map and we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{U}^{an} & \xrightarrow{F^{an}} & \mathbb{A}^{n,an}_K \\ \uparrow p & & \uparrow p \\ \mathcal{U}(L) & \longrightarrow & L^n \end{array}$$

Then by assumption $y \in \mathcal{U}(L)$, and according to lemma 3.1.12,

$$F^{an}(p(y)) = F^{an}(x) = \eta_{0,(r_1,\dots,r_s,1,\dots,1)}.$$

It is a standard fact that $d(\mathcal{H}(\eta_{0,(r_1,\dots,r_s,1,\dots,1)})/K) = n$. Hence since $\dim(\mathcal{U}^{\mathrm{an}}) = n$ (we mean here the dimension as a k-analytic space), for any point u in the fiber $(F^{\mathrm{an}})^{-1}(\eta_{0,(r_1,\dots,r_s,1,\dots,1)})$,

$$d(\mathcal{H}(u)/\mathcal{H}(\eta_{0,(r_1,\dots,r_s,1,\dots,1)})) = 0.$$

According to the formula (3.4), we deduce that $(F^{\mathrm{an}})^{-1}(\eta_{0,(r_1,\dots,r_s,1,\dots,1)})$ is a 0-dimensional $\mathcal{H}(\eta_{0,(r_1,\dots,r_s,1,\dots,1)})$ -analytic space. Hence as a topological space, $(F^{\mathrm{an}})^{-1}(\eta_{0,(r_1,\dots,r_s,1,\dots,1)})$ is discrete (thanks to the definition of the dimension of a k-analytic space [Ber90, p.34]). In particular, x is an isolated point in its fiber $(F^{\mathrm{an}})^{-1}(\eta_{0,(r_1,\dots,r_s,1,\dots,1)})$.

By a simple continuity argument, there exists a neighbourhood V_1 of y in $\mathcal{U}(L)$ such that all points $v \in V_1$ satisfy exactly the same conditions than y listed above. By this we mean in particular that

$$\text{For } i = 1\dots t,\ \left|\frac{P_i(v)}{Q_i(v)}\right| = \left|\frac{P_i(y)}{Q_i(y)}\right|.$$

$$\text{For } i = t+1\dots n,\ \widetilde{\frac{P_i(v)}{Q_i(v)}} = \widetilde{\frac{P_i(y)}{Q_i(y)}}.$$

Hence, the same argument as for y can be applied to v. Namely, for all $v \in V_1$, $F^{an}(p(v)) = \eta_{0,(r_1,\dots,r_t,1,\dots,1)}$. But since $p(y) = x$, p is continuous, and since x is isolated in its fibre $(F^{an})^{-1}(\eta_{0,(r_1,\dots,r_t,1,\dots,1)})$, there exists a neighbourhood $V \subset V_1$ of y such that $p(V) = \{x\}$. □

Theorem 3.1.14. *Let S be a nonempty subanalytic set of $\left((K^{alg})^\circ\right)^n$. The following numbers are equal:*

1. *The greatest d for which there exists a coordinate projection $\pi : (K^{alg^\circ})^n \to (K^{alg^\circ})^d$ such that $\pi(S)$ has nonempty interior.*
2. *The greatest d for which there exists a coordinate projection $\pi : \left((K^{alg})^\circ\right)^n \to \left((K^{alg})^\circ\right)^d$ such that $\pi(S)$ is somewhere dense (that is to say its adherence has nonempty interior).*
3. $$d = \max_{x \in S_{Berko}} tr\ deg(\widetilde{\mathcal{H}(x)}/\widetilde{K}).$$
4. *The greatest d such that there is a non-Archimedean extension $K \to L$ and a point $x \in S(L^{alg})$ such that $tr\ deg(\widetilde{K(x)}/\widetilde{K}) = d$.*
5. $$d = \max_{x \in S_{Berko}} d(\mathcal{H}(x)/K).$$

We define the dimension of S to be the number d which satisfies these equivalent properties. If S is empty we set $\dim(S) = -\infty$.

Proof. Here $(i) \Rightarrow (j)$ will mean that the integer d_i defined in (i) is smaller than d_j, the integer defined in (j).

(1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (4). We can introduce $r := |\lambda| \in |K^*|$ and $c \in \pi(S)$ such that $B(c,r)$, the closed ball of center c and radius r in $\left((K^{\text{alg}})^\circ\right)^d$, is included in $\overline{\pi(S)}$. Let $K^{\text{alg}} \to L$ be a complete non-Archimedean extension with L algebraically closed, and such that there exists $(y_1, \ldots, y_d) \in L^d$ such that $|y_i| = 1$ and the $\widetilde{y_i}$ are algebraically independent over $\widetilde{K^{\text{alg}}}$. Now, according to the uniform quantifier elimination theorem, we still have [1] $B(c,r)_L \subset \overline{\pi(S(L))}$. In particular, $c + \lambda y \in \overline{\pi(S(L))}$. But if we replace the y_i's by very close elements, we will still have that $|y_i| = 1$ and the $\widetilde{y_i}$ are algebraically independent over $\widetilde{K^{\text{alg}}}$. Hence we can in fact assume that $c + \lambda y \in \pi(S(L))$. Hence by definition, we can complete the y_i's, with some $y_{d+1}, \ldots, y_n$, such that $(y_1, \ldots, y_n) \in S(L)$. Now, tr.deg.$((\widetilde{K(c+\lambda y)}/\widetilde{K^{\text{alg}}}) = d$. Hence tr.deg.$(\widetilde{K(c+\lambda y)}/K) = d$ because in the following diagram

$$\begin{array}{ccc} K^{\text{alg}}(c+\lambda y) & \longleftarrow & K^{\text{alg}} \\ \uparrow & & \uparrow \\ K(c+\lambda y) & \longleftarrow & K \end{array}$$

the horizontal inclusions have a residual transcendental degree 0.

(3) $\Leftrightarrow$ (4) follows from the definition of S_{Berko}.

(3) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (1) Let $x \in S_{\text{Berko}}$ such that $d(\mathcal{H}(x)/K) = d$. Then there exists $K \to L$ a complete non-Archimedean algebraically closed extension, and $y \in L^n$ such that $p(y) = x$ where $p : L^n \to \mathbb{B}^n_K$ is as in remark 3.1.1. For instance take $L = \widehat{\mathcal{H}(x)^{alg}}$.

Then $d(K(y)/K) = d(\mathcal{H}(x)/K) = d$, hence there exists $1 \le i_1 < \ldots < i_d \le n$ such that $d(K(y_1, \ldots, y_n)/K) = d(K(y_{i_1}, \ldots, y_{i_d})/K) = d$. Let then $\pi : L^n \to L^d$ be the coordinate projection along the coordinates $i_k, k = 1 \ldots d$ and let $z := (y_{i_1}, \ldots, y_{i_d}) = \pi(y) \in \pi(S)(L)$.

1. If T is a subanalytic set, $\overline{T}$ is also subanalytic because the closure can be defined by a first order formula (see 3.1.24).

Then $\mathrm{d}(K(z)/K) = d$. Then according to proposition 3.1.13, there exists a neighbourhood V of z in L^d such that $p(V) = \{p(z)\}$ where $p : L^d \to \mathbb{B}_K^d$. Since $z \in \pi(S)_{\mathrm{Berko}}$, according to lemma 3.1.6, $V \subset \pi(S)(L)$. Hence $\pi(S)(L)$ contains V, hence has nonempty interior. Since having nonempty interior is definable with a first order formula, according to the uniform quantifier elimination theorem, $\pi(S)$ has also non empty interior. □

Lemma 3.1.15. *The dimension of subanalytic sets is invariant by scalar extension. By this we mean the following. Let $K \to L$ be a non-Archimedean extension, and let S be a subanalytic set of $\left((K^{alg})^\circ\right)^n$, and $S(L^{alg})$ the associated subanalytic set in $((L^{alg})^\circ)^n$. Then* $\dim(S) = \dim(S(L^{alg}))$.

Proof. This follow from the uniform quantifier elimination theorem and the fact that condition (1) in the above theorem can be expressed by a first order formula. □

Proposition 3.1.16. *Let $S \subset \left((K^{alg})^\circ\right)^n$ be a subanalytic set.*

1. $\dim(S) = 0 \Leftrightarrow S$ *is nonempty and finite.*
2. $\dim(S) = n \Leftrightarrow S$ *has non empty interior* $\Leftrightarrow S$ *is somewhere dense.*
3. $\dim(S) < n \Leftrightarrow S$ *has empty interior* $\Leftrightarrow S$ *is nowhere dense.*

Proof.

1. If S is finite, its dimension is clearly 0 according to the characterization (1) of theorem 3.1.14.

   Conversely, if $\dim(S) = 0$, then for $i = 1 \dots n$, let π_i be the coordinate projection along the i-th coordinate $\pi_i : \left((K^{\mathrm{alg}})^\circ\right)^n \to (K^{\mathrm{alg}})^\circ$. Then $\pi_i(S)$ has empty interior. But according to proposition 0.3.19 $\pi_i(S)$ is in fact semianalytic, and it is easy to see that a semi-analytic set of $(K^{\mathrm{alg}})^\circ$ is either finite or has nonempty interior (according to the description of semialgebraic sets of K^{alg} as Swiss cheese). Hence each $\pi_i(S)$ must be finite, so S itself is finite.

2. and 3. are consequences of the characterisations (1) and (2) of the theorem 3.1.14 for $d = n$.

□

Remark 3.1.17. It is tempting to imagine a generalization of (2) above like
" if $S \subset T$ are subanalytic sets, then $\dim(T) = \dim(S) \Leftrightarrow S$ has nonempty interior in T."

Actually, $\Rightarrow$ is true in general [2], but $\Leftarrow$ is false in general [3]. However, if S is an embedded manifold of pure dimension, then this equivalence is true.

Proposition 3.1.18. *If $S_i,\ i = 1 \dots N$ are subanalytic sets of $\left((K^{alg})^\circ\right)^n$,*

$$\dim\left(\bigcup_{i=1}^{N} S_i\right) = \max_{i=1\dots N} \dim(S_i).$$

Proof. This follows from the characterisation (5) in theorem 3.1.14 in term of points in Berkovich spaces. □

2. Take $S = \cup_i S_i$ a stratification of S in disjoint embedded manifolds of pure dimension, $T_i := T \cap S_i$. Then for some i, $\dim(T_i) = \dim(S_i) = \dim(S)$. Since $\dim(\overline{\cup_{j \neq i} S_j} \setminus \cup_{j \neq i} S_j) < \dim(S)$, we can even assume that S_i is open in S. We can also reduce the problem to the case when T is a subanalytic submanifold of pure dimension, and then it is true.
3. Take $S = \{1\} \cup (K^{\mathrm{alg}})^{\circ\circ} \subset (K^{\mathrm{alg}})^\circ$ and $T = \{1\}$.

More generally, let $\{S_i\}_{i\in I}$ be an arbitrary family of subanalytic sets of $\left((K^{\mathrm{alg}})^\circ\right)^n$ and let us assume that $\bigcup_{i\in I}(S_i)_{\mathrm{Berko}}$ is a subanalytic set of $\mathbb{B}^n_K$ then $\dim(\bigcup_{i\in I}(S_i)_{\mathrm{Berko}}) = \max_i \dim((S_i)_{\mathrm{Berko}})$. This property is false if we remove the $_{\mathrm{Berko}}$, as can be seen if we take all the singletons $\{x\}$ of $(K^{\mathrm{alg}})^\circ$ for instance.

Proposition 3.1.19. *Let S be a subanalytic subset of $\left((K^{alg})^\circ\right)^{n+m}$. For $x \in \left((K^{alg})^\circ\right)^n$, let*

$$S_x := \{y \in \left((K^{alg})^\circ\right)^m \mid (x,y) \in S\} = \pi^{-1}(x)$$

where $\pi : \left((K^{alg})^\circ\right)^{n+m} \to \left((K^{alg})^\circ\right)^n$ is the projection on the first n coordinates. For each integer i, let

$$S(i) := \{x \in \left((K^{alg})^\circ\right)^n \mid \dim(S_x) = i\}$$

$$S^{(i)} := \{(x,y) \in S \text{ with } x \in \left((K^{alg})^\circ\right)^n,\ y \in \left((K^{alg})^\circ\right)^m \mid \dim(S_x) = i\} = \pi^{-1}(S(i)).$$

Then $S(i)$ and $S^{(i)}$ are subanalytic sets. Moreover, this is compatible with scalar extension, that is to say, if $K \subset L$ is an extension of complete fields $S(i)(L^{alg}) = S(L^{alg})(i)$ and $S^{(i)}(L^{alg}) = S(L^{alg})^{(i)}$.

Proof. If T is a subanalytic set of $\left((K^{\mathrm{alg}})^\circ\right)^m$, according to the characterisation (1) for the dimension in theorem 3.1.14, the property $\dim(T) \geq d$ is expressible at the first order. Namely, the property "U has non empty interior" can be formulated by the formula :
"there exists a point a point $c \in U$, a radius $\gamma \in \Gamma$ such that U contains the ball of center c and radius γ."
Now $\dim(T) \geq d$ if and only if there exists a coordinate projection $\pi : \left((K^{\mathrm{alg}})^\circ\right)^m \to \left((K^{\mathrm{alg}})^\circ\right)^d$ such that $\pi(T)$ has non empty interior, and since there are only finitely many coordinate projections, the properties $\dim(T) \geq d$ and $\dim(T) = d$ are well definable with a first order formula. This implies that $S(i)$ and hence $S^{(i)}$ are subanalytic. The fact that it behaves well if we increase the field K is then a consequence of the uniform quantifier elimination theorem. ☐

Proposition 3.1.20. *With the above notations,*

$$\dim\left(S^{(i)}\right) = \dim\left(S(i)\right) + i.$$

As a corollary,

$$\dim(S) = \max_{i\geq 0} \dim(S(i)) + i.$$

Proof. Let us prove first that $\dim(S^{(i)}) \leq \dim(S(i)) + i$. To do that we use the characterization (5) of theorem 3.1.14. Let $K \subset L$ be a complete extension, $(x,y) \in S^{(i)}(L^{alg})$. Then $x \in S(i)(L^{alg})$ hence

$$d(K(x)/K) \leq \dim(S(i)) \tag{3.5}$$

Let $M := \widehat{K(x)} \subset \widehat{L^{alg}}$ seen as sub-valued field of L. Then $y \in S_x(L^{alg})$ hence by assumption

$$d(M(y)/M) \leq i = \dim(S_x).$$

Now in the following diagram

$$\begin{array}{ccc} M(x,y) = M(y) & \longleftarrow & K(x,y) \\ \uparrow & & \uparrow \\ M & \longleftarrow & K(x) \end{array}$$

the two horizontal field inclusions satisfy $d(\cdot/\cdot) = 0$, hence

$$d(K(x,y)/K(x)) = d(M(y)/M) \leq i. \tag{3.6}$$

Hence the inequalities (3.5) and (3.6) imply that

$$d(K(x,y)/K) = d(K(x,y)/K(x)) + d(K(x)/K) \leq \dim(S(i)) + i$$

hence $\dim(S^{(i)}) \leq \dim(S(i)) + i$.
Let us prove conversely that $\dim(S(i)) + i \leq \dim(S^{(i)})$.
Let $K \subset L$ be a complete extension and $x \in S(i)(L^{alg})$ such that

$$d(K(x)/K) = \dim(S(i)). \tag{3.7}$$

We can assume that $L = \widehat{K(x)^{alg}}$.
Since $\dim(S_x) = i$, there exists $L \subset M$ a complete extension, and $y \in S_x(M^{alg})$ such that $d(L(y)/L) = i$. Then $(x,y) \in S^{(i)}(M^{alg})$. For the same reason as above,

$$d(K(x,y)/K(x)) = d(L(y)/L) = i \tag{3.8}$$

because $L = \widehat{K(x)^{alg}}$. Hence with (3.7) and (3.8):

$$\dim(S^{(i)}) \geq d(K(x,y)/K) = d(K(x,y)/K(y) + d(K(x)/K) = \dim(S(i)) + i.$$

The corollary now follows from the fact that $S = \cup_i S^{(i)}$ and proposition 3.1.18. □

Proposition 3.1.21. *If S (resp. T) is a subanalytic set of $\left((K^{alg})^\circ\right)^n$ (resp. $\left((K^{alg})^\circ\right)^m$), then $S \times T$ is a subanalytic set of $\left((K^{alg})^\circ\right)^{n+m}$ and*

$$\dim(S \times T) = \dim(S) + \dim(T).$$

Proof. Following [vdD89, 1.5 (i)]
Let $d := \dim(T)$. If we use notations of definition 3.1.19, then by definition

$$(S \times T)^{(i)} = \begin{cases} S \times T & \text{if } i = d \\ \emptyset & \textit{otherwise} \end{cases}$$

and $(S \times T)(d) = S$. Hence according to proposition 3.1.20,

$$\dim(S \times T) = \dim\left((S \times T)^{(d)}\right) = \dim\left((S \times T)(d)\right) + d = \dim(S) + d = \dim(S) + \dim(T).$$

□

Proposition 3.1.22. *Let $f : X \to Y$ be a subanalytic map of subanalytic sets of $\left((K^{alg})^\circ\right)^n$ and $\left((K^{alg})^\circ\right)^m$.*

1. $\dim(f(X)) \leq \dim(X)$.
2. *If f is injective,* $\dim(f(X)) = \dim(X)$.
3. *If f is bijective* $\dim(X) = \dim(Y)$.

Proof. Since 2. and 3. are consequences of 1. so we only have to prove 1. But 1. follows directly from lemma 3.1.9 and the characterisation 5. of the dimension in theorem 3.1.14. □

We give however another proof of this result, following the strategy of [vdD89, 1.5 (ii)]

If $X \subset ((K^{\mathrm{alg}})^\circ)^n$ and $Y \subset ((K^{\mathrm{alg}})^\circ)^m$ are subanalytic, let $G := \mathrm{Graph}(f) \subset ((K^{\mathrm{alg}})^\circ)^{n+m}$ be the graph of f which is by definition a subanalytic set of $((K^{\mathrm{alg}})^\circ)^{n+m}$. We consider

$$\begin{array}{ccc} & G & \\ {}^{p_1}\swarrow\sim & & \sim\searrow^{p_2} \\ X & \xrightarrow{f} & Y \end{array}$$

With the above notations, $G = G^{(0)}$ and $X = G(0)$. Hence according to proposition 3.1.20 with $i = 0$,

$$\dim(G) = \dim(G^{(0)}) = \dim(G(0)) + 0 = \dim(X).$$

Now, according to characterization (1) of the dimension given in theorem 3.1.14 in terms of coordinate projections, it is clear that $\dim(p_2(G)) \leq \dim(G)$ because p_2 is a coordinate projection. Hence

$$\dim(f(X)) = \dim(p_2(G)) \leq \dim(G) = \dim(X).$$

Lemma 3.1.23. *Let X, Y be strictly K-affinoid spaces, $f : X \to Y$ be a morphism of strictly K-affinoid spaces (seen as Berkovich spaces). Let $S_{Berko} \subset X$ be a subanalytic set. Let $y \in Y$ be such that $d(\mathcal{H}(y)/K) = \dim(S_{Berko})$. Then $(f^{-1}(y)) \cap S_{Berko}$ is finite.*

Proof. In this context we can consider the cartesian diagram of Berkovich spaces over K (see [Ber90, p 48]):

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\beta}\uparrow & & \uparrow \\ X_y & \xrightarrow{f'} & \mathcal{M}(\mathcal{H}(y)) \end{array}$$

where X_y is an $\mathcal{H}(y)$-affinoid space for which β induces a homeomorphism $X_y \simeq f^{-1}(y)$. Moreover, if $z \in X_y$, and $x := \beta(z) \in X$, there is a canonical isomorphism of complete fields above K : $\mathcal{H}(z) \simeq \mathcal{H}(x)$. In particular

$$d(\mathcal{H}(x)/K) = d(\mathcal{H}(z)/K) = d(\mathcal{H}(z)/\mathcal{H}(y)) + d(\mathcal{H}(y)/K). \tag{3.9}$$

Now if the subanalytic set S_{Berko} enters the picture we obtain:

$$\begin{array}{ccccc} S_{\mathrm{Berko}} & \hookrightarrow & X & \xrightarrow{f} & Y \\ & & {\scriptstyle\beta}\uparrow & & \uparrow \\ S_y := \beta^{-1}(S_{\mathrm{Berko}}) & \hookrightarrow & X_y & \xrightarrow{f'} & \mathcal{M}(\mathcal{H}(y)) \end{array}$$

We claim that S_y is a subanalytic set of the $\mathcal{H}(y)$-affinoid (Berkovich) space X_y. Indeed, if $\varphi(\underline{x})$ is a quantifier-free $\mathcal{L}_{an}^D$ formula defining $S_{\mathrm{Berko}} \subset X$, then $\varphi(\underline{x}) \wedge (f(\underline{x}) = \underline{y_0})$ is a quantifier-free $\mathcal{L}_{an}^D$-formula which defines $S_y \subset X_y$, where $\underline{y_0}$ is the n-uple of $(\mathcal{H}(y))^n$ canonically attached to y.

Hence if $z \in S_y$, $x := \beta(z)$, since $x \in S_{\mathrm{Berko}}$, $\dim(S_{\mathrm{Berko}}) \geq d(\mathcal{H}(x)/K)$, hence according to (3.9):

$$\dim(S_{\mathrm{Berko}}) \geq d(\mathcal{H}(x)/K) = d(\mathcal{H}(z)/\mathcal{H}(y)) + d(\mathcal{H}(y)/K).$$

Since this is true for all $z \in S_y$, it follows that

$$\dim(S_{\mathrm{Berko}}) \geq \dim(S_y) + d(\mathcal{H}(y)/K).$$

Since we were assuming that $d(\mathcal{H}(y)/K) = \dim(S_{\mathrm{Berko}})$, it follows that $\dim(S_y) = 0$ hence according to proposition 3.1.16 (1), S_y is finite, and this concludes the proof since S_y is in bijection with $(f^{-1}(y)) \cap S_{\mathrm{Berko}}$. $\square$

Proposition 3.1.24. *Let S be a subanalytic set of $\left((K^{alg})^\circ\right)^n$.*

1. *$\overline{S}$ is a subanalytic set and its formation is compatible with complete field extensions $K \to L$:*
$$\overline{S}(L^{alg}) = \overline{S(L^{alg})}.$$
2. *$\dim(\overline{S} \setminus S) < \dim(S)$.*

Proof. 1. Is true because the closure of a set in $\left((K^{\mathrm{alg}})^\circ\right)^n$ can be expressed by a first order formula:

$$\overline{S} = \{x \in \left((K^{\mathrm{alg}})^\circ\right)^n \mid \forall \varepsilon \in \Gamma\ \exists s \in S \text{ such that } |x - s| \leq \varepsilon\}.$$

We now prove 2. Let $d := \dim(S)$, and let us assume that $\dim((\overline{S} \setminus S) \geq d$.
Using the characterizations 5. of theorem 3.1.14, there must exist some coordinate projection

$$\pi : \left((K^{\mathrm{alg}})^\circ\right)^n \to \left((K^{\mathrm{alg}})^\circ\right)^d$$

and $x \in \left(\overline{S}\right)_{\mathrm{Berko}} \setminus S_{\mathrm{Berko}}$, such that if we set $y = \pi_{\mathrm{Berko}}(x)$, then $d(\mathcal{H}(y)/K) = d$.
Let now $K \to L$ be a complete extension with L algebraically closed and let us consider the commutative diagram:

$$\begin{array}{ccc}
x \in \mathbb{B}^n_K & \xrightarrow{\pi_{\mathrm{Berko}}} & \mathbb{B}^d_K \ni y \\
{\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle p_2} \\
z \in (L^\circ)^n & \xrightarrow{\pi_L} & (L^\circ)^d \ni w
\end{array}$$

where $z \in (L^\circ)^n$ is a preimage of x by p_1, and $w = \pi_L(z)$. Hence $(z_1, \dots, z_n) \in \overline{S}(L) \setminus S(L)$.
Furthermore, $p_2(w) = y$ and y is an Abhyankar point of $\mathbb{B}^d_K$ (i.e. $d(\mathcal{H}(y)/K) = d$), hence according to proposition 3.1.13, there exists V a neighbourhood of w in $(L^\circ)^d$ such that $p_2(V) = \{y\}$.
Since π_L is continuous, there exists W a neighbourhood of z in $(L^\circ)^n$ such that $\pi_L(W) \subset V$, which implies that

$$p_2(\pi_L(W)) = \pi_{\mathrm{Berko}}(p_1(W)) = \{y\}. \tag{3.10}$$

Since $d(\mathcal{H}(y)/K) = \dim(S_{\mathrm{Berko}}) = d$, according to lemma 3.1.23, $S_{\mathrm{Berko}} \cap ((\pi_{\mathrm{Berko}})^{-1}(y))$ is finite and does not contain x because $x \in \overline{S}_{\mathrm{Berko}} \setminus S_{\mathrm{Berko}}$. Since $\mathbb{B}^n_K$ is a Hausdorff space and p_1 is continuous, using (3.10), we can shrink W the neighbourhood of $z \in (L^\circ)^n$ in such a way that

$$p_1(W) \cap \left(S_{\mathrm{Berko}} \cap (\pi_{\mathrm{Berko}})^{-1}(y)\right)) = \emptyset. \tag{3.11}$$

But by assumption, since W is a neighbourhood of z and $z \in \overline{S(L)}$, the intersection $W \cap S(L)$ is nonempty. Now if $u \in W \cap S(L)$, then $p_1(u) \in p_1(W)$ and $p_1(u) \in S_{\mathrm{Berko}} \cap ((\pi_{\mathrm{Berko}})^{-1}(y))$ which is a contradiction according to (3.11).

$\square$

Corollary 3.1.25. *Let S be a nonempty subanalytic subset of $\big((K^{alg})^\circ\big)^n$, and let $T \subset S$ be a subanalytic subset of S. If* $\dim(T) = \dim(S)$ *then T has nonempty interior in S.*

Proof. If we set
$$U := S \setminus T$$
we obtain the decomposition
$$S = U \coprod T.$$
Hence,
$$T = (T \setminus \overline{U}) \coprod (T \cap (\overline{U} \setminus U)).$$
Then according to proposition 3.1.24 (2),
$$\dim(T \cap (\overline{U} \setminus U)) \leq \dim(\overline{U} \setminus U) < \dim(U) \leq \dim(T).$$
So, $\dim(T \setminus \overline{U}) = \dim(T) \geq 0$. In particular, $T \setminus \overline{U}$ is a nonempty open subset of S included in T, hence T has nonempty interior in S. □

Remark 3.1.26. If $S \subset \big((K^{\mathrm{alg}})^\circ\big)^n$ is a a subanalytic set we want to stress out that in general,
$$(\overline{S})_{\mathrm{Berko}} \neq \overline{(S_{\mathrm{Berko}})},$$
where the first adherence is taken in $\big((K^{\mathrm{alg}})^\circ\big)^n$, and the second in $\mathbb{B}^n_K$.

Actually it is true that $(\overline{S})_{\mathrm{Berko}} \subset \overline{(S_{\mathrm{Berko}})}$ but the converse is false. For instance, if $S = (K^{\mathrm{alg}})^{\circ\circ}$, then $\overline{S} = S = (K^{\mathrm{alg}})^{\circ\circ}$, whereas $S_{\mathrm{Berko}} = \{x \in \mathbb{B}^1_K \mid |X(x)| < 1\}$ and $\overline{S_{\mathrm{Berko}}} = S_{\mathrm{Berko}} \cup \{\eta\}$ where η is the Gauss point of $\mathbb{B}_K$.

Proposition 3.1.27. *Let $I = (f_1, \dots, f_p)$ be an ideal of $S_{m,n}$. Then $V(I)$ defines a subset of $\big((K^{alg})^\circ\big)^m \times \big((K^{alg})^{\circ\circ}\big)^n$, hence also a subset of $\big((K^{alg})^\circ\big)^{m+n}$ which is subanalytic. Let us consider the quasi-affinoid algebra $A := S_{m,n}/I$, and let $\mathcal{X} :=$ Spec(A). Then*
$$\dim(V(I)) = \dim(\mathcal{X}).$$

Proof. By definition, $V(I)$ can be identified with $X = \mathrm{Sp}(A)$, the rigid K-space associated to A, as defined in section 0.5.2.

Step 1. First, if $n = 0$, that is to say if X is a K-affinoid space, then the result follows from the characterisation 5. we have given of $\dim(V(I))$ and the fact that the dimension of X as a K-analytic space is the Krull dimension of A [Ber90, p.34].

Step 2. If $n > 0$, let us denote by $\{A_j\}_{j\geq 1}$ the K-affinoid algebras such that $X = \cup_{j \geq 1} X_j$ where $X_j = \mathrm{Sp}(A_j)$ is an increasing sequence of affinoid domains of X (notations of section 0.5.2). Then $\dim(X) = \max_{j\geq 1} \dim(X_j)$, for instance because X is the union of its affinoid domains X_j, and using the characterization 5. od dimension in theorem 3.1.14. It then follows that
$$\begin{aligned}
\dim(\mathcal{X}) &= \max_{x \in \mathrm{Max}(A)} \big(\dim(\mathcal{O}_{\mathcal{X},x})\big) \\
&= \max_{x \in \mathrm{Max}(A)} \big(\dim(\widehat{\mathcal{O}_{\mathcal{X},x}})\big) \\
&= \max_{x \in X} \big(\dim(\widehat{\mathcal{O}_{X,x}})\big) && (3.12) \\
&= \max_{x \in X} \big(\dim(\mathcal{O}_{X,x})\big) && (3.13) \\
&= \dim(X).
\end{aligned}$$

The equality (3.12) follows from the Nullstellensatz for quasi-affinoid algebras [LR00d, 4.1.1] and from [LR00d, 4.2.1]. The equality (3.13) is a classical fact from rigid geometry [BGR84, 7.3.2.3]. □

Remark 3.1.28. More generally, we can define the dimension of subanalytic sets of a quasi-affinoid variety (see definition 0.5.17). The invariance of dimension by subanalytic bijection (proposition 3.1.22 (3)) is necessary to prove that this definition works well. This theory of dimension then satisfies all the properties mentioned above.

3.1.3 Smooth stratification theorem

In this subsection, we want to state the *Smooth stratification theorem* [LR00a, Theorem 4.4] following the exposition of [LR00a]. We will need this result in the next subsection.

If F is a complete non-Archimedean field, there is a classical notion of **F-analytic manifold**, which mimics the definition in the Archimedean case : an F-analytic manifold is obtained by gluing some open subsets of F^n with bianalytic maps. If X is an F-analytic manifold, we can talk about the dimension of X at a point x, denoted by $\dim_{\mathrm{manif},x}(X)$, and then define the dimension $\dim_{\mathrm{manif}}(X)$ of an F-analytic manifold as the supremum of $\dim_{\mathrm{manif},x}(X)$ for $x \in X$. We can talk about a submanifold of an F-analytic manifold. A morphism between F-analytic manifolds is defined as in the Archimedean case: it is locally given by analytic maps in the charts. We can talk about a submanifold of $(F^\circ)^n$. We refer to [Ser06, Chapter III] for a general discussion of F-analytic manifolds, and to [Ser65] for a nice classification result when F is a local field.

Definition 3.1.29 ([LR00a, 1.4])**.** We say that a subanalytic set $S \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ is a **subanalytic embedded K^{alg}-analytic manifold** if the subanalytic set $X(\widehat{K^{\mathrm{alg}}}) \subset \left((\widehat{K^{\mathrm{alg}}})^\circ\right)^n$ is a submanifold of $\left((\widehat{K^{\mathrm{alg}}})^\circ\right)^n$ whose transition functions can be locally given by power series with coefficients in K^{alg}.

We will not use the last condition on the coefficients of the transition functions.

Theorem 3.1.30 ([LR00a, 4.4])**.** *1. Let $S \subset \left((K^{alg})^\circ\right)^n$ be a subanalytic set. There exists a decomposition*

$$S = S_1 \cup \ldots \cup S_l \tag{3.14}$$

where the S_i's are pairwise disjoint subanalytic embedded K^{alg}-analytic manifolds.

2. If $T \subset \left((K^{alg})^\circ\right)^n$ is an subanalytic embedded K^{alg}-analytic manifold, then $\dim(T) = \dim_{manif}\left(T(\widehat{K^{alg}})\right)$*. In particular, the equality* (3.14) *implies that*

$$\dim(S) = \max_{i=1\ldots l}\Big(\dim_{manif}\big((S_i(\widehat{K^{alg}})\Big).$$

Remark 3.1.31. We want to stress the fact that for a subanalytic embedded K^{alg}-analytic manifold $S \subset \left((K^{\mathrm{alg}})^\circ\right)^n$, the equality of the dimension of S as defined in 3.1.14 (1) and the dimension of $S(\widehat{K^{\mathrm{alg}}})$ as a $\widehat{K^{\mathrm{alg}}}$-analytic manifold is more subtle that it might seem. The part 2. of the above theorem states that this is true when S is subanalytic, but this is false if we remove the subanalytic assumption.

Indeed, let K be a non-Archimedean and algebraically closed field such that

$$\mathrm{Card}(\widetilde{K}) = \mathrm{Card}((K^{\mathrm{alg}})^\circ). \tag{3.15}$$

For instance $K = \widehat{\mathbb{C}((t))^{alg}}$ satisfies these conditions. Let then $\alpha : \widetilde{K} \to (K^{\mathrm{alg}})^\circ$ be a bijection (a surjection would work actually), and let $\beta : \widetilde{K} \to (K^{\mathrm{alg}})^\circ$ be a section of the reduction map. Let then

$$V := \{(\alpha(\lambda), \beta(\lambda)) \mid \lambda \in \widetilde{K}\}.$$

Since by construction the family $\{\beta(\lambda)\}_{\lambda \in \widetilde{K}}$ is discrete, V is an infinite discrete union of singletons, hence a K-analytic submanifold of $\left((K^{\mathrm{alg}})^\circ\right)^2$ of pure dimension 0. However, if $\pi : \left((K^{\mathrm{alg}})^\circ\right)^2 \to (K^{\mathrm{alg}})^\circ$ is the first projection, $\pi(V) = (K^{\mathrm{alg}})^\circ$. In this example, the projection of a submanifold of dimension 0 has dimension 1.

This illustrates the fact that subanalytic submanifolds behave well, and that V is not subanalytic.

With the same kind of idea, it is also possible to find $S \subset \left((K^{\mathrm{alg}})^\circ\right)^2$ a submanifold of analytic dimension 0 such that the analytic dimension of $\overline{S}$ is 1. Take

$$\{S = (x, \pi^n \beta(\alpha^{-1}(x)))\}_{x \in (K^{\mathrm{alg}})^\circ, n \in \mathbb{N}}$$

Then S is a submanifold of dimension 0, and contains $(K^{\mathrm{alg}})^\circ \times \{0\}$ in its adherence.

We can now give more characterizations of the dimension :

Proposition 3.1.32. *Let $S \subset \left((K^{alg})^\circ\right)^n$ be a subanalytic set. Then $d = \dim(X)$ is also characterized as*

1. $$d_1 = \max_{i=1\ldots N} \dim_{manif}(S_i)$$

   where $S = \cup_{i=1\ldots N} S_i$ is a decomposition of S in subanalytic embedded K^{alg}-analytic manifolds. Moreover d_1 is independent of the decomposition.

2. *The greatest integer d_2 such that there exists a subanalytic map $\left((K^{alg})^\circ\right)^{d_2} \to S$ defined over some finite extension of K which is injective.*

3. *The greatest d_3 such that there exists a subanalytic map $f : S \to \Gamma^{d_3}$ such that $f(X)$ has non empty interior.*

Proof. The equality $d = d_1$ was contained in theorem 3.1.30.

$d_2 \leq d$ is a consequence of proposition 3.1.22.

$d_2 \geq d$. According to the smooth stratification theorem, S contains a subanalytic embedded K^{alg}-analytic manifold $T \subset S$ such that $\dim_{\mathrm{manif}}(T) = \dim(S)$. Hence if $x \in T$, some judicious coordinate projection $\pi : \left((K^{\mathrm{alg}})^\circ\right)^n \to \left((K^{\mathrm{alg}})^\circ\right)^d$ will induce some bijection between some open neighbourhood of x in T and some closed ball around $\pi(x)$ in $\left((K^{\mathrm{alg}})^\circ\right)^d$. Taking the inverse of it defines the required injective map.

$d = d_3$. In one hand, $d \leq d_3$ because we can easily find some surjective map $f : S \to \left((K^{\mathrm{alg}})^\circ\right)^d$, and then composing with a norm function yields the desired function. The converse inequality $d \geq d_3$ is a consequence of theorem 4.2.21 which will be proved in the next chapter. ☐

3.1.4 Continuity of subanalytic maps

In the Archimedean setting, Osgood's theorem asserts that if Ω is an open subset of $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}^n$ an injective holomorphic map, then $f(\Omega)$ is open and f induces a biholomorphic map between Ω and $f(\Omega)$. See [Nar71, th.5 p.86] or [Ran86, Th. 2.14].

Proposition 3.1.33. *Let K be a complete algebraically closed non-Archimedean field. Let $f : X \to Y$ be a morphism of K-analytic manifolds of the same pure dimension d such that f is injective. Then $f(X)$ is open in Y and the restriction $X \xrightarrow{f_{|X}} f(X)$ is a homeomorphism.*

This proposition is false if K is not algebraically closed. In characteristic p for instance, the Frobenius map $x \to x^p$ is an analytic endomorphism of K, which is precisely open if and only if K is perfect.

Proof. Step 1. It is equivalent to prove that f is open because then, the map $f_{|X} : (K^\circ)^n \to \mathrm{im}(f)$ is bijective, continuous and open, hence a homeomorphism.

Step 2. To prove this we can work locally, hence restrict to an injective analytic map $f : (K^\circ)^n \to (K^\circ)^n$.

Step 3. For all $x \in (K^\circ)^n$, and $y = f(x)$ we must show that for all neighbourhood V of x, there exists $W \subset V$ another open neighbourhood of x such that $f(W)$ is open (this would show that f is open).

Since closed balls around x form a basis of neighbourhood of x, it is sufficient to prove that if $f : (K^\circ)^n \to (K^\circ)^n$ is an injective analytic map and $x \in (K^\circ)^n$, there exists an open neighbourhood V of x such that $f(V)$ is open.

Step 4. By definition of an analytic map, reducing [4] the source and the target of f again, we can assume that $f : (K^\circ)^n \to (K^\circ)^n$ is given by n analytic functions. Hence, f is the restriction to the set of rigid points (here the K-points) of a morphism of k-affinoid spaces $f_{\text{Berko}} : \mathbb{B}^n_K \to \mathbb{B}^n_K$.

Step 5. Since the restriction of f_{Berko} to the set of rigid points is f, hence injective, and since $x \in (\mathbb{B}^n_K)_{\text{rig}}$ by assumption, $\dim_x(f_{\text{Berko}}) = 0$. Hence according to [Duc07, Th 3.2], there exists an analytic neighbourhood V of x in $\mathbb{B}^n_K$ such that f_{Berko} factorizes as

$$V \xrightarrow{\alpha} W \xrightarrow{\beta} T \xrightarrow{\gamma} \mathbb{B}^n_K$$

where α is finite, β is the immersion of an analytic domain of T and γ is étale.

Step 6. If $z := \beta(\alpha(x))$, since it is a rigid point, γ is a local homeomorphism at z. Hence if we replace $\mathbb{B}^n$ by a small enough ball containing z, we can assume that $T = \mathbb{B}^n$. Hence f_{Berko} factorizes now as

$$V \xrightarrow{\alpha} W \xrightarrow{\beta} \mathbb{B}^n$$

with α finite and β the immersion of an analytic domain.

Step 7. Now if we replace (in the target) $\mathbb{B}^n_K$ by a ball small enough that contains y and is included in W, we can even assume that f_{Berko} factorizes as

$$V \xrightarrow{\alpha} \mathbb{B}^n_K$$

with α finite.

Step 8. Now remember that α is still injective at the level of the rigid points, and that V_{rig} is a neighbourhood of x in $(K^\circ)^n$. Since α is finite, $\alpha(V)$ is a Zariski-closed subset of $\mathbb{B}^n_K$. On the other hand, according to proposition 3.1.22 (*ii*), since $\dim(V) = n$ and since α_{rig} is injective, $\dim(\alpha(V)) = n$. If $\alpha(V)$ was a proper Zariski-closed subset of $\mathbb{B}^n_K$, we would have $\dim(\alpha(V)) < n$ according to lemma 3.1.27 which is impossible. Hence $\alpha(V) = \mathbb{B}^n_K$ which is a neighbourhood of y. □

4. If we were on $\mathbb{C}$, we would not need to do that.

Proposition 3.1.34. *Let $f : X \to Y$ be a subanalytic map between subanalytic sets $X \subset \left((K^{alg})^\circ\right)^n$ and $Y \subset \left((K^{alg})^\circ\right)^m$. Then there exists a decomposition $X = X_1 \cup \ldots \cup X_N$ in subanalytic embedded K^{alg}-analytic manifolds such that for all i $f_{|X_i}$ is continuous.*

Proof. We prove this by induction on $\dim(X)$.

If $\dim(X) = 0$, X is finite (proposition 3.1.16 (1)) and the result is clear.

If $d = \dim(X) > 0$, we take a decomposition $X = X_1 \cup \ldots \cup X_N$ in subanalytic embedded K^{alg}-analytic manifolds (theorem 3.1.30). We can then replace f by the various $f_{|X_i} : X_i \to Y$. Since $\dim(X_i) \leq \dim(X)$ we can replace X by X_i and hence assume that X is a subanalytic embedded K^{alg}-analytic manifold (if it happens that $\dim(X_i) < \dim(X)$ we can already apply the induction hypothesis). We then have the following commutative diagram:

$$\begin{array}{ccc} & G = \mathrm{Graph}(f) & \\ {}^{p_1}\swarrow_{\sim} & & \searrow^{p_2} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

where G is a subanalytic set of $((K^{\mathrm{alg}})^\circ)^{n+m}$. We apply again the smooth stratification theorem to G and obtain a decomposition $G = G_1 \cup \ldots \cup G_N$ where each G_i is a subanalytic embedded K^{alg}-analytic manifold. Since $p_1 : G \to X$ is a subanalytic map which is bijective, thanks to proposition 3.1.22, $\dim(G) = \dim(X)$. Hence, $\dim(G_i) \leq \dim(X)$ for all i. Let us set $X_i := p_1(G_i)$.

If $\dim(G_i) < \dim(X)$, then $\dim(X_i) < \dim(X)$ so by induction hypothesis, we can again stratify X_i in $X'_{i,j}s$ such that $f_{|X_{i,j}}$ is continuous.

If $\dim(X_i) = \dim(X)$, then $p_{1|G_i} : G_i \to X$ is an injective subanalytic map. Hence $(p_1)_{|G_i})_{\widehat{K^{\mathrm{alg}}}} : G_i(\widehat{K^{\mathrm{alg}}}) \to X(\widehat{K^{\mathrm{alg}}})$ is an injective analytic map between $\widehat{K^{\mathrm{alg}}}$-analytic manifolds of the same dimension. Hence according to proposition 3.1.33, it induces a homeomorphism between $G_i(\widehat{K^{\mathrm{alg}}})$ and $X_i(\widehat{K^{\mathrm{alg}}})$. Hence $p_{1|G_i}$ also induces a homeomorphism between G_i and X_i. Eventually, $f_{|X_i} = p_2 \circ (p_{1|G_i})^{-1}$ is then continuous. □

Corollary 3.1.35. *Let $f : X \to Y$ be a subanalytic map between subanalytic sets $X \subset \left((K^{alg})^\circ\right)^n$ and $Y \subset \left((K^{alg})^\circ\right)^m$. Let*

$$D := \{x \in X \mid f \textit{ is not continuous at } x\}.$$

Then D is a subanalytic set and

$$\dim(D) < \dim(X).$$

Proof. First D is subanalytic because continuity at x can be expressed by a first order formula.

Then, let us assume that $\dim(D) = \dim(X)$. According to corollary 3.1.25, D has nonempty interior in X, so replacing X by $\overset{\circ}{D}$ (which is subanalytic), we can assume that $D = X$, i.e. that f is nowhere continuous.

But according to proposition 3.1.34, there is a decomposition in subanalytic sets $X = X_1 \cup \ldots \cup X_n$ such that for all i, $f_{|X_i}$ is continuous. There must exist some i such that $\dim(X_i) = \dim(X)$, so according to corollary 3.1.25 again, X_i has nonempty interior in X, so if $x \in \overset{\circ}{X_i}$, f is continuous at x which contradicts the fact that f is nowhere continuous. □

Proposition 3.1.36. *Let $X \subset \left((K^{alg})^\circ\right)^m \times \Gamma^n$ be a subanalytic set, and $f : X \to \Gamma$ a subanalytic map. Then there exists a decomposition $X = X_1 \cup \ldots \cup X_N$ in subanalytic sets such that for all i $f_{|X_i} : X_i \to \Gamma$ is continuous.*
Actually, it is even defined as

$$f_{|X_i} = \sqrt[a_i]{|g_i(x)|\alpha^{u_i}}$$

where g_i is a continuous subanalytic map $U_i \to K^{alg}$ for some subanalytic $U_i \subset \left((K^{alg})^\circ\right)^m$, and $u_i \in \mathbb{Z}^n$, $a_i \in \mathbb{N}^*$. We denote by $x = (x_1, \ldots, x_m) \in \left((K^{alg})^\circ\right)^m$ the field variables, and by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma^n$ the value group variables.

Proof. Let $G := \text{Graph}(f) \subset \left((K^{\text{alg}})^\circ\right)^m \times \Gamma^n \times \Gamma$ which is subanalytic. We will use the variable x (resp. α, γ) for the variable of $\left((K^{\text{alg}})^\circ\right)^m$ (resp. Γ^n, Γ). It is enough to consider the case where G is a finite intersection of subsets defined by inequalities

$$|F(x)|\gamma^a \bowtie |G(x)|\alpha^u$$

where $a \in \mathbb{N}$, $u \in \mathbb{Z}^n$, $\bowtie \in \{=, >, <\}$ and $F, G : \left((K^{\text{alg}})^\circ\right)^m \to K^{alg}$ are D-functions. We can then restrict to subanalytic sets where $F(x)$ and $G(x) \neq 0$, hence their quotient $\frac{F}{G}$ will be D-functions $\left((K^{\text{alg}})^\circ\right)^m \to K^{alg}$. Hence G appears as a finite union of subsets of the form

$$I_j = \{(x, \alpha, \gamma) \in \left((K^{\text{alg}})^\circ\right)^m \times \Gamma^n \times \Gamma \mid |g_j(x)|\alpha^{u_j} < \gamma^{b_j} < |h_j(x)|\alpha^{v_j}\} \; j = 1 \ldots M$$

and

$$\{(x, \alpha, \gamma) \in \left((K^{\text{alg}})^\circ\right)^m \times \Gamma^n \times \Gamma \mid |f_i(x)|\alpha^{u_i} = \gamma^{a_i}\} \; i = 1 \ldots N$$

plus some additional conditions involving only the variables (x, α).
Since by assumption, for all $(x, \alpha) \in X$, $G_{(x,\alpha)} = \{\gamma \in \Gamma \mid (x, \alpha, \gamma) \in G\}$ is a singleton, we can in fact remove the intervals I_j. We then set temporarily

$$X_i = \{(x, \alpha) \in X \mid f(x, \alpha) = \sqrt[a_i]{|f_i(x)|\alpha^{u_i}}\}.$$

We then have

$$f_{|X_i}(x, \alpha) = \sqrt[a_i]{|f_i(x)|\alpha^{u_i}}$$

with $f_i : \left((K^{\text{alg}})^\circ\right)^n \to K^{alg}$ an D-function. Now if we apply proposition 3.1.34 to the map $f_i : \left((K^{\text{alg}})^\circ\right)^n \to K^{alg}$ we know that we can shrink the $X_i's$ so that the map

$$\begin{array}{ccc} X_i & \to & K^{alg} \\ (x, \alpha) & \mapsto & f_i(x) \end{array}$$

is continuous. As a consequence,

$$\begin{array}{ccc} X_i & \to & \Gamma \\ (x, \alpha) & \mapsto & |f_i(x)| \end{array}$$

is continuous, and then

$$f_{|X_i} : \begin{array}{ccc} X_i & \to & \Gamma \\ (x, \alpha) & \mapsto & \sqrt[a_i]{|f_i(x)|\alpha^{u_i}} \end{array}$$

is also continuous. □

3.2 Dimension in ACVF

In this part K is a model of ACVF. We want to explain (following closely results of [vdD89]) how a natural dimension for definable subsets of K^n can be defined.

Lemma 3.2.1. *[Duc12a, lemma 1.1] Let $\mathcal{X}$ be an affine irreducible K-scheme of finite type, U a nonempty open subset of $\mathcal{X}(K)$.*
Then for all nonempty Zariski-open subset $\mathcal{V}$ of $\mathcal{X}$, $U \cap \mathcal{V}(K) \neq \emptyset$.
In other words, U is Zariski-dense in $\mathcal{X}$.
In other words again, if $\mathcal{Z}$ is a proper Zariski closed subset of $\mathcal{X}$, $\mathcal{Z}(K)$ is nowhere dense.

Proposition 3.2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an étale morphism of K-schemes of finite type. Then $f_K : \mathcal{X}(K) \to \mathcal{Y}(K)$ is a local homeomorphism.*

In fact, it is sufficient to assume that K is a Henselian valued field, as will be clear from the proof. Although this proposition is a very well known fact, we haven't found an explicit reference for it.

Proof. This is a local property on $\mathcal{X}$. Let then $x \in \mathcal{X}(K)$, $y = f(x) \in \mathcal{Y}(K)$. According to the local description of étale morphisms, we can assume that $\mathcal{Y} = \mathrm{Spec}(A)$ is affine, that there exists $P \in A[T]$, such that if we set

$$\mathcal{Z} = \mathrm{Spec}(A[T, P'(T)^{-1}]/(P(T))$$

and g is the standard étale morphism $g : \mathcal{Z} \to \mathcal{Y}$, then f factorises as

$$\mathcal{X} \xrightarrow{j} \mathcal{Z} \xrightarrow{g} \mathcal{Y}$$

where j is an open immersion. Here again since the property is local on $\mathcal{X}$, we can even assume that $\mathcal{X} = \mathrm{Spec}(A[T, P'(T)^{-1}]/(P(T))$ and that that f is the standard étale morphism $f : \mathrm{Spec}(A[T, P'(T)^{-1}]/(P(T)) \to \mathrm{Spec}(A)$.
Hence

$$\mathcal{X}(K) = \{(y, t) \in \mathcal{Y}(K) \times K \mid P(y, t) = 0 \text{ and } P'(y, t) \neq 0\}.$$

After a change of variable, we can assume that

$$x = (y, 0) \tag{3.16}$$

New we write $P(T) = a_0 + a_1 T + \ldots + a_n T^n$ where the $a_i' s \in A$. Now (3.16) implies that $a_0(y) = 0$ and the condition $P'(y, 0) \neq 0$ that $a_1(y) \neq 0$.
For $\lambda \in K^*$, we set

$$G(T) = P(\lambda T) = a_0 + \lambda a_1 T + \cdots + \lambda^n a_n T^n.$$

For $|\lambda|$ small enough, since $a_1(y) \neq 0$, we will have $|\lambda a_1(y)| > |\lambda^i a_i(y)|$ for $i \neq 1$, so that replacing P by G, we can assume that $|a_1(y)| > |a_i(y)|, i \neq 1$. Moreover, by continuity of the function $a_i' s$, there exists a neighbourhood of y, $V \subset \mathcal{Y}(K)$ such that for all $z \in V$, $|a_1(z)| > |a_i(z)|, i \neq 1$ and $|a_1(z) - a_1(y)| < |a_1(y)|$.
Hence, dividing P by $a_1(y)$, we can assume that $P = \sum a_i T^i$ with

i) $a_1(y) = 1$.

ii) For all $z \in V$, $|a_1(z) - 1| < 1$.

iii) For all $z \in V$, and $i \neq 0$, $|a_1(z)| = 1 > |a_i(z)|$

In particular, for all $z \in V$, $P(z,T) = \sum a_i(z)T^i \in (K^{\mathrm{alg}})^\circ[T]$ and $\widetilde{P(z,T)} = T$. According to Hensel's lemma, there exists a unique $t_z \in (K^{\mathrm{alg}})^{\circ\circ}$ such that $P(z,t_z) = 0$. In addition,

$$P'(z,t_z) = a_1(z) + 2a_2(z)t_z^2 + \ldots + na_n(z)t_z^{n-1},$$

and by assumption, $|a_1(z)| > |2a_2(z)t_z^2 + \ldots + na_n(z)t_z^{n-1}|$ hence $P'(z,t_z) \neq 0$, so $(z,t_z) \in \mathcal{X}(K)$. Hence if we set $U := \{(z,t) \in V \times (K^{\mathrm{alg}})^\circ \mid P(z,t) = 0\}$, this is an open neighbourhood of x in $\mathcal{X}(K)$, and the restriction map

$$\begin{array}{cccc} f_{|U}: & U & \to & V \\ & (z,t) & \mapsto & z \end{array}$$

is a continuous bijection. We only have to prove that it is a homeomorphism. Let then $\varepsilon \leq 1 \in |K^*|$. There exists a neighbourhood W of y such that for $z \in W$, $|a_0(z)| < \varepsilon$. Hence $|t_z| < \varepsilon$, otherwise, we would have, $|a_1(z)t_z| = |t_z| > |a_i(z)t_z^i|$ for $i \neq 1$, in contradiction with $P(z,t_z) = 0$. This proves that $f_{|U}^{-1}$ is continuous at y. Now, if $y' = f(x') \in V$, we make the same argument as above, and it follows that $f_{|U}^{-1}$ is also continuous at y'. Hence $f_{|U}$ is a homeomorphism. ☐

Let $\mathcal{X}$ be a K-scheme of finite type and let $U \subset \mathcal{X}(K)$. Since K is algebraically closed, we can identify U with a subset of $\mathcal{X}$ and we denote by $\dim_{Zar}(U)$ the Zariski dimension of the Zariski closure of U in $\mathcal{X}$:

$$\dim_{Zar}(U) := \dim(\overline{U}^{\mathrm{Zar}}).$$

Lemma 3.2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of integral K-schemes of finite type, $U \subset \mathcal{X}(K)$ a nonempty open subset.*

1. *If f is dominant,* $\dim_{Zar}(f(U)) = \dim_{Zar}(\mathcal{Y})$.
2. $\dim_{Zar}(f(U)) \leq \dim_{Zar}(U)$.
3. *If $f_{|U}$ is injective,* $\dim_{Zar}(f(U)) = \dim_{Zar}(\mathcal{X}) = \dim_{Zar}(U)$.

Proof. 1. Otherwise, we would have $f(U) \subset \mathcal{Z}$ a proper Zariski closed subset of $\mathcal{Y}$, and since f is dominant, $f^{-1}(\mathcal{Z})$ would be a proper Zariski closed subset of $\mathcal{X}$ which contains U, and this contradicts lemma 3.2.1.

2. $\dim_{Zar}(f(U)) \leq \dim_{Zar}(f(\mathcal{X})) \leq \dim_{Zar}(\mathcal{X}) = \dim_{Zar}(U)$, the last equality being lemma 3.2.1.
3. Replacing $\mathcal{Y}$ by $\overline{f(X)}$, we can assume that f is dominant. If we denote by η the generic point of $\mathcal{Y}$, then $\mathcal{X}_\eta$ is an irreducible variety of dimension δ say. Then $\mathcal{C}$, the set of points of x such that $\dim_x(f) = \delta$ is a constructible set of $\mathcal{X}$ which contains η', the generic point of $\mathcal{X}$. Hence, $\mathcal{C}$ contains $\mathcal{V}$ some nonempty open set of $\mathcal{X}$. Let us pick some $x \in \mathcal{V}(K) \cap U$ (this is possible thanks to lemma 3.2.1) and let us set $y := f(x)$. Then $\mathcal{X}_y \cap U$ is a nonempty open subset of $\mathcal{X}_y(K)$, which is supposed to be the singleton $\{x\}$ because $f_{|U}$ is injective. But since $\dim_x(\mathcal{X}_y) = \delta$, thanks to lemma 3.2.1 again, this implies that $\delta = 0$. So $\dim(\mathcal{X}) = \dim(\mathcal{Y})$, and according to 1., $\dim_{Zar}(f(U)) = \dim(\mathcal{Y}) = \dim(\mathcal{X})$.

☐

Definition 3.2.4. Let $\mathcal{X}$ be an affine K-scheme of finite type. We will say that $U \subset \mathcal{X}(K)$ is a special open subset if it is a finite intersection:

$$U = \Big(\bigcap_{i=1}^{M} \{x \in \mathcal{X}(K) \mid |f_i(x)| < |F_i(x)|\}\Big) \cap \Big(\bigcap_{j=1}^{N} \{x \in \mathcal{X}(K) \mid |g_j(x)| \leq |G_j(x)| \neq 0\}\Big)$$

where $f_i, F_i, g_j, G_j \in \mathcal{O}(\mathcal{X})$.

Theorem 3.2.5. *Let d be an integer, and S be a nonempty definable subset of K^n. Then the following definitions of the integer d are equivalent:*

1. *There exists a decomposition $S = \cup_{i=1}^m S_i$ where for each i, $S_i \neq \emptyset$, $S_i = \mathcal{X}_i \cap U_i$ where $\mathcal{X}_i$ is an irreducible variety of (algebraic) dimension d_i and U_i is a special open subset of K^n, such that $d = \max_i d_i$. This number is independent of the choice of the decomposition.*
2. $d = \dim_{Zar}(\overline{S}^{Zar})$.
3. $$d = \max_{\substack{x \in S(L) \\ K \subset L}} tr.deg.(K(x)/K)$$
   where $K \subset L$ describes the set of extensions of algebraically closed valued fields of K.
4. *The greatest integer d such that there exists a coordinate projection $\pi : K^n \to K^d$ such that $\pi(S)$ is somewhere dense (that is to say, its closure has nonempty interior).*
5. *The greatest integer d such that there exists a coordinate projection $\pi : K^n \to K^d$ such that $\pi(S)$ has nonempty interior.*
6. *The greatest integer d such that there exists an injective definable map $f : (K^\circ)^d \to S$.*
7. *The greatest integer d such that there exists an injective definable map $f : (K^\circ)^d \to S$ which is a homeomorphism on its image (in particular f is continuous).*
8. *The greatest integer d such that there exists an open subset of S, $U \subset S$ and a map defined by polynomials (according to the embedding of U in K^n) $f : U \to (K^\circ)^d$ which is a homeomorphism.*

This integer d is called the dimension of S, denoted by $\dim(S)$.

Proof. We will denote by d_i the integer defined by the i-th definition in the above list.

- $d_1 = d_2$: Let $S = \cup_i S_i$ as in (1). Then according to lemma 3.2.1, for each i, $\overline{S_i}^{Zar} = \mathcal{X}_i$. Hence $\overline{S}^{Zar} = \cup_i \overline{S_i}^{Zar} = \cup_i \mathcal{X}_i$, so $\dim_{Zar}(\overline{S}^{Zar}) = d_1$.
- $d_1 \leq d_3$: Let us assume that S contains a definable set $T = \mathcal{X} \cap U$ where $\mathcal{X}$ is an integral subvariety of Zariski dimension d_1 and U a nonempty special open subset of $\mathcal{X}(K)$.
  Then for each $f \in K[\mathcal{X}] \setminus \{0\}$, let us consider the first order formula $\varphi_f(x) =$ "$x \in T(K)$ and $f(x) \neq 0$".
  According to lemma 3.2.1, for each $f_1, \ldots, f_m \in K[\mathcal{X}] \setminus \{0\}$, the formula $\varphi_{f_1} \wedge \ldots \wedge \varphi_{f_m}$ is satisfiable. Hence if L is an algebraically closed valued field which is an extension of K and which is Card(K)-saturated, there exists $x \in L^n$ which satisfies all the φ_f for $f \in K[\mathcal{X}] \setminus \{0\}$. The point of the scheme $\mathcal{X}$ that is associated to $x \in \mathcal{X}(L)$ is then the generic point of $\mathcal{X}$, hence tr deg$(K(x)/K) = \dim_{Zar}(\mathcal{X}) = d_1$.
- $d_1 \geq d_3$: Let $S = \cup_{i=1\ldots m} S_i$ where $S_i = \mathcal{X}_i \cap U_i$, and let L be an algebraically closed valued extension of K, $x \in S(L)$, such that tr deg$(K(x)/k) = d_3$. Then $x \in \mathcal{X}_i(L)$ for some i, hence $d_3 =$ tr deg$(K(x)/K) \leq \dim_{Zar}(X_i) \leq d_1$.
- $d_1 \geq d_4$: Let $\pi : K^n \to K^{d_4}$ be a coordinate projection such that $\overline{\pi(S)}$ has nonempty interior, and let $S = \cup_i (\mathcal{X}_i \cap U_i)$ be a decomposition of S as in (1) (such a decomposition always exists). Then $\pi(S) \subset \cup_i \pi(\mathcal{X}_i)$, which is a constructible subset whose Zariski-dimension is less than $d_1 = \max_i \dim_{Zar}(\mathcal{X}_i)$. Since $\overline{\pi(S)}$ has nonempty interior, lemma 3.2.1 implies that $d_1 \geq d_4$.
- $d_4 \geq d_5$ is clear.

- $d_5 \geq d_1$: We can assume that $S \supset \mathcal{X} \cap U$ where $\mathcal{X}$ is irreducible of Zariski-dimension d_1, and U a nonempty subset of $\mathcal{X}(K)$. Then there exists a coordinate projection $\pi : \mathbb{A}^n_K \to \mathbb{A}^{d_1}_K$ such that its restriction $\pi : \mathcal{X} \to \mathbb{A}^{d_1}_K$ is dominant. Write $\pi(S) = \cup_i(\mathcal{Z}_i \cap U_i)$ with $\mathcal{Z}_i$ an irreducible subvariety of $\mathbb{A}^{d_1}_K$ of dimension δ_i and U_i a nonempty special open subset of K^n. If one of the δ_i's is d_1 (in which case $\mathcal{Z}_i = \mathbb{A}^{d_1}$). Then $U_i \cap \mathcal{Z}_i = U_i$ and indeed $d_5 \geq d_1$. Otherwise, $\pi(S) \subset \mathcal{Z}$ where $\mathcal{Z}$ is a proper Zariski closed subset of $\mathbb{A}^{d_1}$, then since π is dominant, $\pi^{-1}(\mathcal{Z})$ is a proper Zariski closed subset of $\mathcal{X}$ which contains S, and this contradicts lemma 3.2.1.
- $d_1 = d_2 \geq d_6$ is a consequence of lemma 3.2.3 (3).
- $d_6 \geq d_7$ is clear.
- $d_7 \geq d_8$: If $U \subset S$ is a definable subset and $f : U \to ((K^{\mathrm{alg}})^\circ)^{d_8}$ an homeomorphism as in (8), then f^{-1} is as in (7) so $d_7 \geq d_8$
- $d_8 \geq d_1$: If S contains some nonempty $\mathcal{X} \cap U$ where $\mathcal{X}$ is an irreducible subvariety of dimension d_1 and U an open subset of $\mathcal{X}(K)$, then thanks to lemma 3.2.1 we can replace $\mathcal{X}$ by a Zariski open subset and assume that $\mathcal{X}$ is smooth over $\mathrm{Spec}(K)$ of dimension d_1. Since $x \in \mathcal{X}(K) \cap U$, there exists a Zariski-open neighbourhood of x, $\mathcal{U}$, and $f : \mathcal{U} \to \mathbb{A}^{d_1}$ an étale morphism. Hence according to proposition, 3.2.2, there exists V an open neighbourhood of x and W an open neighbourhood of $f(x)$ such that f induces an homeomorphism between V and W. After shrinking it, and a linear change of variables in W, we can then assume that $W = (K^\circ)^{d_1}$.

□

We now want to give some general properties of the dimension of definable subsets of K^m.

Proposition 3.2.6. *If $S_i,\ i = 1 \dots N$ are definable sets of K^n,*

$$\dim(\cup_i S_i) = \max_i \dim(S_i).$$

Proof. Follows from the characterisation 3. of dim in 3.2.5. □

Proposition 3.2.7. *Let S be a definable subset of K^{n+m}. For $x \in K^n$, let $S_x := \{y \in K^m \mid (x, y) \in S\} = \pi^{-1}(x)$ where $\pi : K^{n+m} \to K^n$ is the projection on the first coordinates. For each integer i, let*

$$S(i) = \{x \in K^n \mid \dim(S_x) = i\}$$

$$S^{(i)} = \{(x, y) \in S \mid \dim(S_x) = i\} = \pi^{-1}(S(i)).$$

Then $S(i)$ and $S^{(i)}$ are definable subsets. Moreover, if $K \subset L$ where L is a model of ACVF, $S(i)(L) = S(L)(i)$ and $S^{(i)}(L) = S(L)^{(i)}$.

Proof. If T is a definable subset of K^m, according to the fifth characterisation of the dimension in theorem 3.2.5, the property $\dim(S) \geq d$ is expressible by a first order formula. Namely it is equivalent to
There exists a coordinate projection $\pi : K^m \to K^d$, a point $c \in K^d$, a radius $\gamma \in \Gamma$ such that $\pi(T)$ contains the ball of center c and radius γ.

Since there are only finitely many coordinate projections, this is indeed expressible with a first order formula. This implies the definability of $S(i)$ and hence of $S^{(i)}$. The fact that it behaves well if we increase the field K is then a consequence of the quantifier elimination in ACVF.

□

Proposition 3.2.8. *With the above notations,*

$$\dim(S^{(i)}) = \dim(S(i)) + i.$$

Proof. Let us prove that $\dim(S^{(i)}) \leq \dim(S(i)) + i$. Let $K \subset L$ be an extension, $(x, y) \in S^{(i)}(L)$. Then $x \in S(i)(L)$ hence

$$\text{tr.deg.}(K(x)/K) \leq \dim(S(i)) \tag{3.17}$$

Let $M := K(x)^{alg} \subset L$ seen as a sub-valued field of L. Then $y \in S_x(L)$ hence by assumption tr deg $(M(y)/M) \leq i = \dim(S_x)$. Now in the following diagram

$$\begin{array}{ccc} M(x,y) = M(y) & \longleftarrow & K(x,y) \\ \uparrow & & \uparrow \\ M & \longleftarrow & K(x) \end{array}$$

the two horizontal field inclusions are algebraic extensions, hence

$$\text{tr.deg.}(M(y)/M) = \text{tr.deg.}(K(x,y)/K(x)) \leq i \tag{3.18}$$

Hence the inequalities (3.17) and (3.18) imply that $\text{tr.deg.}(K(x, y)/K) \leq \dim(S(i)) + i$, hence $\dim(S^{(i)}) \leq \dim(S(i)) + i$.
Let us prove conversely that $\dim(S(i)) + i \leq \dim(S^{(i)})$. Let $K \subset L$ an extension of ACVF and $x \in S(i)(L)$ such that $\text{tr.deg.}(K(x)/K) = \dim(S(i))$. We can then assume that $L = K(x)^{alg}$. Then $\dim(S_x) = i$ so there exists $L \subset M$ an extension of ACVF and $y \in S_x(M)$ such that $\text{tr.deg.}(L(y)/L) = i$. Then $(x, y) \in S^{(i)}(M)$. For the same reason as above, $\text{tr.deg.}(K(x, y)/K(x)) = \text{tr.deg.}(L(y)/L)$, hence $\dim(S^{(i)}) \geq \text{tr.deg.}(K(x, y)/K) = \dim(S(i)) + i$. □

Proposition 3.2.9. *If S (resp. T) is a definable subset of K^n (resp. K^m), then* $\dim(S \times T) = \dim(S) + \dim(T)$.

Proof. This is a consequence of the fact that if $\mathcal{X}$ and $\mathcal{Y}$ are K-varieties, $\dim_{Zar}(\mathcal{X} \times_K \mathcal{Y}) = \dim_{Zar}(\mathcal{X}) + \dim_{Zar}(\mathcal{Y})$, and of the characterisation (1) in theorem 3.2.5.

Using the last proposition we could also have proved it in a similar way as proposition 3.1.21. □

Proposition 3.2.10. *Let $f : X \to Y$ be a morphism of definable sets.*

1. $\dim(f(X)) \leq \dim(X)$.
2. *If f is injective,* $\dim(f(X)) = \dim(X)$.
3. *If f is bijective* $\dim(X) = \dim(Y)$.

Proof. In fact we have already proved (1) in the proof of theorem 3.2.5. We give another proof.
If $X \subset K^n$ and $Y \subset K^m$ are definable subsets, let $G := \text{Graph}(f) \subset K^{n+m}$ be the graph of f. By definition, it is a definable subset.

$$\begin{array}{ccccc} & & G & & \\ & \overset{p_1}{\swarrow}\sim & & \sim\overset{p_2}{\searrow} & \\ X & & \xrightarrow{\quad f \quad} & & Y \end{array}$$

But $G = G^{(0)}$ and $X = G(0)$. Hence $\dim(G) = \dim(G^{(0)}) = \dim(G(0)) + 0 = \dim(X)$. Finally, since p_2 is an algebraic morphism, we have that

$$\dim(f(X)) = \dim p_2(G) \leq \dim(G) = \dim(X)$$

as a consequence of lemma 3.2.3. Another way to see it, is to see G as a subset $K^n \times K^m$, reversing the first n and last m coordinates. Hence

$$G = \cup_i G^{(i)},\ \dim(G) = \max_i \dim(G^{(i)}) = \max_i \dim(G(i) + i).$$

Hence

$$\dim(G) \geq \max_i \dim(G(i)) = \dim(\cup_{i=0}^m G(i)) = \dim(p_2(G)) = \dim(f(X))$$

□

Proposition 3.2.11. *Let S be a definable subset of K^n.*

1. *$\overline{S}$ is a definable subset. Moreover, for all extension of ACVF $K \subset L$, $\overline{S}(L) = \overline{S(L)}$.*
2. $\dim(\overline{S} \setminus S) < \dim(S)$.

Proof.

By definition,

$$\overline{S} = \{x \in K^n \mid \forall \varepsilon \in \Gamma\ \exists s \in S \text{ such that } |x - s| \leq \varepsilon\},$$

which proves that $\overline{S}$ is definable.

According to the definition (1) of dimension in theorem 3.2.5, we can assume that $S = \mathcal{X} \cap U$ is nonempty where $\mathcal{X}$ is an irreducible Zariski-closed subset, and U is a special open subset of K^n of the form:

$$U = \bigcap_i \{x \in K^n \mid |f_i(x)| < |F_i(x)|\} \bigcap_j \{x \in K^n \mid |g_j(x)| \leq |G_j(x)| \neq 0\}$$

where $f_i, F_i, g_j, G_j \in K[X]$. In addition, we can assume that the $(G_j)_{|\mathcal{X}}$'s $\in K[\mathcal{X}] \setminus \{0\}$. Now

$$\overline{\{x \in \mathcal{X} \mid |f_i(x)| < |F_i(x)|\}} \subset \\ \{x \in \mathcal{X} \mid |f_i(x)| < |F_i(x)|\} \cup \{x \in \mathcal{X} \mid f_i(x) = 0 \text{ or } F_i(x) = 0\}$$

Moreover

$$\overline{\{x \in \mathcal{X} \mid |g_j(x)| \leq |G_j(x)| \neq 0\}} \subset \{x \in \mathcal{X} \mid |g_j(x)| \leq |G_j(x)|\} \\ = \{x \in \mathcal{X} \mid |g_j(x)| \leq |G_j(x)| \neq 0\} \cup \{x \in \mathcal{X} \mid g_j(x) = G_j(x) = 0\}$$

But $\{x \in \mathcal{X} \mid G_j(x) = 0\}$ is a proper subvariety of $\mathcal{X}$ whose dimension is strictly less than $\dim_{Zar}(\mathcal{X})$. Now,

$$\overline{S} \subset \bigcap_i \Big(\{x \in \mathcal{X} \mid |f_i(x)| < |F_i(x)|\} \cup \{x \in \mathcal{X} \mid f_i(x) = 0 \text{ or } F_i(x) = 0\}\Big) \\ \bigcap_j \{x \in \mathcal{X} \mid |g_j(x)| \leq |G_j(x)|\} \\ = \Big(\bigcap_i [\{x \in \mathcal{X} \mid |f_i(x)| < |F_i(x)|\} \cup \{x \in \mathcal{X} \mid f_i(x) = 0 \text{ or } F_i(x) = 0\}]\Big) \\ \bigcap_j \Big[\{x \in \mathcal{X} \mid |g_j(x)| \leq |G_j(x)| \neq 0\} \bigcup \{x \in \mathcal{X} \mid g_j(x) = G_j(x) = 0\}\Big]$$

If we develop the $\bigcup$ with respect the $\bigcap_i$, and $\bigcap_j$ we obtain some sets, one of them being S, and the other ones being contained in some $\{x \in \mathcal{X} \mid g_j(x) = G_j(x) = 0\} \subset \mathcal{X} \cap \{x \in K^n \mid G_j(x) = 0\}$. But, since $\mathcal{X}$ is irreducible, either $\{x \in \mathcal{X} \mid G_j(x) = 0\}$ is a proper Zariski-closed subset of $\mathcal{X}$ and we are done. Or $\{x \in \mathcal{X} \mid G_j(x) = 0\} = \mathcal{X}$, which then contradicts the fact that $\mathcal{X} \cap U \neq \emptyset$. Hence $\dim(\overline{S} \setminus S) < \dim(\mathcal{X})$. $\square$

Chapter 4

Mixed dimension

4.1 Introduction

4.1.1 Motivations from Tropical Geometry

Let K be a non-Archimedean algebraically closed field, Γ its value group (we set $|\cdot| : K \to \Gamma \cup \{0\}$ its norm). The process of tropicalization starts with an object X defined over K, (for instance an algebraic subvariety of $\mathbb{G}^n_{m,K}$), and associates to it a simpler combinatorial object defined over Γ. This process relies on the *tropicalization map*: Trop where

$$\begin{array}{rccc} \text{Trop}: & (K^*)^n & \to & \Gamma^n \\ & (x_1, \dots, x_n) & \mapsto & (|x_1|, \dots, |x_n|). \end{array}$$

For instance when X is a subvariety of $\mathbb{G}^n_{m,K}$, we want to study $\text{Trop}(X)$ which is a subset of Γ^n. Some properties of X reflect in $\text{Trop}(X)$. In the context of k-analytic spaces we will also denote indifferently by Trop the maps

$$\begin{array}{rcl} \text{Trop}: (\mathbb{G}^n_{m,k})^{\text{an}} & \to & (\mathbb{R}^*_+)^n \\ x & \mapsto & (|T_1(x)|, \dots, |T_n(x)|) \end{array}$$

and

$$\begin{array}{rcl} \text{Trop}: \mathbb{A}^{n,an}_k & \to & \mathbb{R}^n_+ \\ x & \mapsto & (|T_1(x)|, \dots, |T_n(x)|) \end{array}$$

where the T_i's are the coordinate functions.

We will say that a set $S \subset (\mathbb{R}^*_+)^n$ is a **polyhedron** if it is a finite intersection of sets of the form

$$\{(\gamma_1, \dots, \gamma_n) \in (\mathbb{R}^*_+)^n \mid \lambda \prod_{i=1}^n \gamma_i^{a_i} \leq \prod_{i=1}^n \gamma_i^{b_i}\}$$

where $a_i, b_i \in \mathbb{N}$ and $\lambda \in \mathbb{R}^*_+$. We say that S is a Γ-rational polyhedron if the λ's belong to Γ. A **polytope** is a compact polyhedron. A **polyhedral set** is a finite union of polyhedra. In these definitions, one can replace $\mathbb{R}^*_+$ by Δ where Δ is an ordered abelian group.

We state some known results:

Theorem. [BG84, Theorem A], [EKL06, 2.2.3] Let X be an irreducible subvariety of $\mathbb{G}^n_{m,K}$ of dimension d. Then $\text{Trop}(X(K)) \subset \Gamma^n$ is a connected polyhedral set of pure dimension d.

Theorem. [Gub07, Th. 1.1] Let X be an irreducible closed analytic [1] subvariety of $(\mathbb{G}_{m,K}^n)^{\text{an}}$ of dimension d. Then $\text{Trop}(X) \subset (\mathbb{R}_+^*)^n$ is a locally finite union of (Γ-rational) d-dimensional polytopes.

Theorem. [Duc12a, Th 3.2] Let X be a compact K-analytic space of dimension d and $f : X \to (\mathbb{G}_{m,K}^n)^{\text{an}}$ an analytic map. Then $\text{Trop}(X) := |f|(X) \subset (\mathbb{R}_+^*)^n$ is a finite union of (Γ-rational) polytopes of dimension $\leq d$.

The initial motivation of this work was to obtain the following generalization of Ducros's result [Duc12a, Theorem 3.2]:

Theorem. 4.3.1 Let X be a compact K-analytic space of dimension d, $f : X \to \mathbb{A}_K^{n,an}$ be an analytic map. Then $|f|(X) \cap (\mathbb{R}_+^*)^n$ is a Γ-rational polyhedral set of $(\mathbb{R}_+^*)^n$ of dimension $\leq d$.

The reader interested by this result can easily proceed to its proof. Let us mention that Ducros's result [Duc12a, Th 3.2] has been motivated by the recent development of a theory of differential forms on Berkovich spaces [CD12].

4.1.2 Definable sets

To prove this we use the language $\mathcal{L}_{an}^D$ introduced by L. Lipshitz in [Lip93]. In fact, in the above theorem 4.3.1, we can prove that $|f|(X)$ is a polyhedral set as a direct application of the quantifier elimination theorem (see 4.3.1) proved by Lipshitz [Lip93, theorem 3.8.2]. But the bound on the dimension in theorem 4.3.1 does not follow obviously [2].

However, the list of results mentionned above [BG84, Gub07, Duc12a] which all contain a bound on the dimension of $\text{Trop}(X)$ was a strong evidence in favour of the bound in theorem 4.3.1. Once one tries to prove it, one realizes that we have a set $X \subset K^m$, a map $f : X \to \Gamma^n$ and we would like to say that

$$\dim(f(X)) \leq \dim(X). \tag{4.1}$$

This inequality should be satisfied in any good theory of dimension. However, here the situation is quite particular, because in (4.1), we are dealing with two different kinds of dimension. On the left side, this a dimension for subsets of K^n, and on the right side, this is the very combinatorial dimension of a polyhedral set of $\mathbb{R}^n$ (or Γ^n in general). What formula (4.1) suggests is that these two dimensions interact nicely. It is then natural to wonder whether it is possible to obtain a unified theory of dimension, where (4.1) would become a simple corollary.

More precisely, assume that we could assign a dimension $\dim(S) \in \mathbb{N}$ to each subsets S of $K^m \times \mathbb{R}^n$ such that

DIMENSION AXIOMS	
Dim1	When $S \subset K^m, \dim(S)$is the *classical* dimension of S as a subset of K^m.
Dim2	When $S \subset \mathbb{R}^n, \dim(S)$ is the *classical* dimension of S as a subset of $\mathbb{R}^n$.
Dim3	If $S \subset K^m \times \mathbb{R}^n, T \subset K^{m'} \times \mathbb{R}^{n'}$ and $f : S \to T$ is a definable map, then $\dim(f(S)) \leq \dim(S)$
Dim4	With S and T as above, $\dim(S \times T) = \dim(S) + \dim(T)$

(4.2)

1. here we assume that $\Gamma \subset \mathbb{R}$ and that K is complete.

2. We want to mention that once one knows that $|f|(X)$ is a finite union of polyhedra, the bound on the dimension is then a corollary of Ducros's result [Duc12a, 3.2]. However, we have tried to obtain a complete proof in an analytic language for aesthetic reasons.

Then (4.1) would be a simple consequence of this dimension theory. Of course, in such a generality this is an unattainable goal. In order to build such a theory, one has to work with a *restricted class* of subsets of $K^m \times \mathbb{R}^n$. In fact, since we want to keep track of the parameters, we will work with a base field K which is valued but not necessarily algebraically closed, and we will consider an extension of the valuation to K^{alg}, and we will set $\Gamma = |(K^{alg})^*|$. There are two *restricted classes* that we will consider in this text:

- The class of definable subsets of $S \subset (K^{alg})^m \times \Gamma^n$, in the first order theory of algebraically valued fields defined over K[3].
- If K is non-Archimedean, the class of subanalytic sets $S \subset \left((K^{\mathrm{alg}})^{\circ}\right)^m \times \Gamma^n$.

One of the main results of this chapter is that for these two classes we can build a good dimension theory, as described in (4.2):

Theorem. 4.2.21 There exists a dimension theory which satisfies the axioms listed in (4.2) when we restrict to:

- definable subsets $S \subset (K^{alg})^m \times \Gamma^n$ in the theory of algebraically closed valued fields, and K^{alg} is a model of ACVF, and we restrict to definable maps $f : S \to T$.
- When $S \subset \left((K^{\mathrm{alg}})^{\circ}\right)^m \times \Gamma^n$ is a subanalytic set, and K is a complete non-Archimedean field, and we restrict to subanalytic maps $f : S \to T$.

Theorem 4.3.1 is then a consequence of the second part of this result.

This result is indeed a way to unify the dimension theory of the two sorts K^{alg} and Γ. It appears however that assigning a dimension $\dim(S) \in \mathbb{N}$ to a set $S \subset (K^{alg})^m \times \Gamma^n$ is quite coarse.

Example 4.1.1. For instance we must have $\dim((K^{alg})^*) = \dim(\Gamma) = 1$. Moreover, the map $|\cdot| : (K^{alg})^* \to \Gamma$ is surjective, which strengthens the fact that $(K^{alg})^*$ and Γ should have the same dimension. We can remark by the way that this map contradicts the intuitive formula : $\dim(\mathrm{im}) + \dim(\mathit{fibers}) = \dim(\mathit{source})$. But at the opposite, if $f : \Gamma \to K^{alg}$ is a definable map (either in the theory of algebraically closed valued fields, or in the theory of subanalytic sets), then it must have a finite image (see lemma 4.2.12), hence $\dim(f(\Gamma)) = 0$.

This example illustrates a general philosophy that we can sum up in:

$$\text{dimension of } K^{alg} \text{ is stronger than dimension of } \Gamma. \tag{4.3}$$

Once we have this idea in mind, let us look at a second example:

Example 4.1.2. Let $S = K \coprod \Gamma^2$. If one wants to give a precise sense to this, one can take

$$S = (\{x\} \times \Gamma^2) \coprod (K \times \{(\gamma_1, \gamma_2)\}) \subset K \times \Gamma^2$$

with $x \in K$ and $(\gamma_1, \gamma_2) \in \Gamma^2$. According to the dimension theory built in theorem 4.2.21, $\dim(S) = 2$. However, one can check that there is no definable map $f : S \to K^n$ such that $\dim(f(S)) = 2$. Hence in that case, saying that S has dimension 2 seems to hide a part of the situation.

More generally (still following the dimension theory mentioned in theorem 4.2.21), we should have $\dim((K^{alg})^m \times \Gamma^n) = m + n$. Now, there exist some definable maps $f : (K^{alg})^m \times \Gamma^n \to \Gamma^{m+n}$ whose image has dimension $m + n$, but there does not exists a definable map $g : (K^{alg})^m \times \Gamma^n \to K^{m+n}$ whose image has dimension $m+n$. Actually, for such a map, it is natural to conjecture that $\dim(\mathrm{im}(g)) \leq m$. This is true, and will follow for instance from theorem 4.2.18.

3. In that case K is an arbitrary algebraically closed valued field, so Γ can be an arbitrary totally ordered abelian divisible group, and the valuation on K is not necessarily of height 1.

Hence we should refine the leitmotiv given in (4.3) : *dimension of K is stronger than dimension of* Γ, and replace it by

| one dimension of K can lead to one dimension of K or one dimension of Γ | and

| one dimension of Γ can only lead to one dimension of Γ. |

In order to take this into account, it then seems that one should look for a dimension theory that would be two-dimensional: to a subset $S \subset (K^{alg})^m \times \Gamma^n$ we would assign a dimension $\dim(S) = (d_1, d_2) \in \mathbb{N}^2$ with d_1 for the part of the valued field K and d_2 for the valued group Γ. In addition, if such a $\mathbb{N}^2$-valued dimension theory would exist, the condition Dim 3 in our list of axioms (4.2) should be replaced by the condition:

if $S \subset (K^{alg})^m \times \Gamma^n$, $\dim(S) = (d_1, d_2)$ and $f : S \to T$ is a definable map, then

$$\dim(f(S)) \leq \max_{i=0..d_1} (d_1 - i, d_2 + i).$$

However this does not completely solve our problems. Indeed, what should be the dimension of $S = K \coprod \Gamma^2$ that we saw in example 4.1.2? We should hesitate between $(1, 0)$ and $(0, 2)$. Now according to the above leitmotiv, it would be unfair to throw $(1, 0)$, but on the other hand since $0 + 2 > 1 + 0$, we do not want neither to throw $(0, 2)$. The only solution seems to keep both of them, and to define the dimension of a definable subset of $(K^{alg})^m \times \Gamma^n$ as a subset of $\mathbb{N}^2$.

In order to obtain a simple statement, we must specify a few things. On $\mathbb{N}^2$ we will consider the partial order $\leq$ where

$$(a, b) \leq (a', b') \Leftrightarrow a \leq b \text{ and } a' \leq b'.$$

We will say that a set $A \subset \mathbb{N}^2$ is a **lower set** if it is stable under $\leq$.

If $(a, b) \in \mathbb{N}^2$, we set

$$\langle (a, b) \rangle := \{(x, y) \in \mathbb{N}^2 \mid (x, y) \leq (a, b)\}.$$

This is the smallest lower set of $\mathbb{N}^2$ containing (a, b).

For aesthetic reasons, if $A, B \subset \mathbb{N}^2$ we will set $\max(A, B) := A \cup B$. With these notations, we then prove :

Theorem. 4.2.18 Let K be a non-Archimedean valued field and let us consider the class of subsets $S \subset (K^{alg})^m \times \Gamma^n$ definable in ACVF (resp. the class of subanalytic sets, $S \subset \left((K^{\text{alg}})^\circ\right)^m \times \Gamma^n$ defined in the language $\mathcal{L}_{an}^D$). Then there exists a dimension function which assigns to each definable (resp. subanalytic) set S a dimension $\dim(S) \subset \mathbb{N}^2$ which is a finite lower set of $\mathbb{N}^2$, satisfying the following properties:

(i) If $S \subset \Gamma^n$, then $\dim(S) = \langle (0, d) \rangle$ where d is the classical dimension of S as a subset of Γ^n as defined in section 4.1.4.

(ii) If $S \subset K^m$ is definable in ACVF (resp. $S \subset \left((K^{\text{alg}})^\circ\right)^m$ is subanalytic), then $\dim(S) = \langle (d, 0) \rangle$ where d is the classical dimension of S as a definable subset in ACVF (resp. subanalytic set) of $(K^{alg})^m$ (resp. $\left((K^{\text{alg}})^\circ\right)^m$) as exposed in section 3.1 (resp. section 3.2).

(iii) If $f : S \to T$ is a definable (resp. subanalytic) map, then [4]

$$\dim(f(S)) \leq \max_{k \geq 0} \big(\dim(S) + (-k, k)\big).$$

4. We use the convention that if $A \subset \mathbb{N}^2$ and $B \subset \mathbb{Z}^2$ then $A + B := (A +_{\mathbb{Z}^2} B) \cap \mathbb{N}^2$ where $A +_{\mathbb{Z}^2} B$ is the usual sum of two subsets $A, B \subset \mathbb{Z}^2$.

(iv) If $f : S \to T$ is a definable (resp. subanalytic) bijection, then $\dim(S) = \dim(T)$.

(v) $\dim(S \times T) = \dim(S) + \dim(T)$.

We want to point out that this theorem implies theorem 4.2.21. Indeed, let us denote by dim the dimension theory valued in subsets of $\mathbb{N}^2$ alluded to in theorem 4.2.18. Then, for a set $S \subset (K^{alg})^m \times \Gamma^n$ definable in ACVF (resp. a subanalytic set $S \subset \left((K^{\mathrm{alg}})^\circ\right)^m \times \Gamma^n$), if we set

$$\dim_{\mathbb{N}}(S) := \max_{(d_1,d_2)\in\dim(S)} d_1 + d_2$$

the properties listed in theorem 4.2.18 above imply that $\dim_{\mathbb{N}}$ fulfils the properties of theorem 4.2.21. This is actually the way we do things: we first prove theorem 4.2.18, and obtain theorem 4.2.21 as a corollary.

4.1.3 Cells

Let us now explain the method used to define the mixed dimension with value in finite lower subsets of $\mathbb{N}^2$ and how we prove theorem 4.2.18. First one has to keep in mind that for definable subsets $S \subset \Gamma^n$, one way to define and study the dimension of S, is to use the cell decomposition theorem [vdD98, Chapter 3]. What does this theorem say? If $(i_1, \ldots, i_n) \in \{0,1\}^n$ one defines inductively what it means for $C \subset \Gamma^n$ to be an $(i_1, \ldots, i_n)$-cell: a (0)-cell (resp. (1)-cell) is a singleton (resp. an open interval) in Γ. Then if $C \subset \Gamma^n$ is a $(i_1, \ldots, i_n)$-cell, an $(i_1, \ldots, i_n, 0)$-cell (resp. $(i_1, \ldots, i_n, 1)$-cell) is a subset $\{(x,\gamma) \in C \times \Gamma \mid \gamma = f(x)\}$ (resp. $\{(x,\gamma) \in C \times \Gamma \mid f(x) < \gamma < g(x)\}$) where $f : C \to \Gamma$ is a continuous definable function (resp. $f, g : C \to \Gamma$ are continuous definable functions such that $f < g$). Roughly speaking, an $(i_1, \ldots, i_n)$-cell is build from graphs (for the 0's appearing in $i_1, \ldots, i_n$) and open domains delimited by functions $f < g$ (for the 1's which appear in $i_1, \ldots, i_n$). The dimension of an $(i_1, \ldots, i_n)$-cell is $i_1 + \ldots + i_n$. The cell decomposition asserts that any definable set $S \subset \Gamma^n$ can be partitioned into finitely many cells $\{C_j\}_{j=1\ldots N}$, and the dimension of S can be defined as the maximum of the dimension of the S_j's. Alternatively,

$$\dim(S) = \max_{\substack{C \text{ an } (i_1,\ldots,i_n)\text{-cell} \\ C\subset S}} i_1 + \ldots + i_n. \tag{4.4}$$

In our situation, if $S \subset (K^{alg})^m \times \Gamma^n$ is definable, for each $x \in (K^{alg})^m$, the fiber of S above x is a definable set $S_x \subset \Gamma^n$, hence can be decomposed into finitely many cells. The idea is to prove that the cell decomposition behaves nicely when x moves. To formalize this we introduce the notion of a cell $C \subset (K^{alg})^m \times \Gamma^n$. So to speak they are *relative cells*. If $X \subset (K^{alg})^m$ is a definable subset, we define what it means for $C \subset X \times \Gamma^n$ to be a $X - (i_1, \ldots, i_n)$ cell (definition 4.2.2), which is a continuous definable family of $(i_1, \ldots, i_n)$-cell (in the above sense) parametrized by X. Then we do things in close analogy with what already exists for cells of Γ^n: we prove a cell decomposition theorem (theorem 4.2.8) for definable subsets of $(K^{alg})^m \times \Gamma^n$, then we define the dimension of a $X - (i_1, \ldots, i_n)$-cell to be $(\dim(X), i_1 + \ldots + i_n) \in \mathbb{N}^2$, and eventually, if $S \subset (K^{alg})^m \times \Gamma^n$ is definable, we define

$$\dim(S) = \{\dim(C) \mid \ C \subset S \text{ is a cell}\}.$$

As a result, $\dim(S)$ is a finite lower set of $\mathbb{N}^2$. This dimension theory satisfies the properties mentioned in theorem 4.2.18. Once we have proved the (mixed) cell decomposition theorem (theorem 4.2.18) the last non-trivial part of theorem 4.2.18 is *(iii)*. To handle it, we reduce to the simple situation of lemma 4.2.16, where we use the simple behaviour of definable maps $f : K \to \Gamma$.

As we have explained in sections 3.1, 3.2, the dimension theories of definable subsets $X \subset K^n$ in ACVF and subanalytic sets $X \subset \left((K^{\mathrm{alg}})^\circ\right)^n$ share many properties. As a consequence, the results exposed in section 4.2, especially concerning the mixed cell decomposition, work in the frameworks of ACVF and subanalytic sets without difference. In 4.1.4 we briefly expose the dimension theory of definable sets $S \subset \Gamma^n$ when Γ is a totally ordered abelian divisible group, referring to [vdD98] [5]. In section 4.2 the proofs are general enough to apply both to definable sets in ACVF and to subanalytic sets. Hence in this part, *definable set* will mean definable in ACVF or subanalytic. In 4.2.1, we prove the mixed cell decomposition theorem for definable sets $S \subset (K^{alg})^m \times \Gamma^n$. Sections 4.2.2 and 4.2.3 are devoted to the proof of the general properties of the mixed dimension of definable subsets $S \subset (K^{alg})^m \times \Gamma^n$ that we have announced.
In 4.3 we explain how this relates to previous works.

A word of warning

In the proofs which will follow, we will always work with variables $\gamma \in \Gamma$. However, it may happen that for some $x \in K^m$, $|f(x)| = 0$. For instance the formula $|0| = \gamma$ does not have any sense if we want to quantify on a variable $\gamma \in \Gamma$. A correct formulation is to work with $\Gamma_0 = \Gamma \cup \{0\}$. For instance, in all the constructions which will follow, at some point of the proofs, we should have distinguished between the cases $f(x) = 0$ and $f(x) \neq 0$ when we consider (definable) functions $f : K^m \to K$. However, these two sets are definable, and on the first one, $|f(x)| = 0$ is constant. On the other one, $f(x) \neq 0$, hence $|f(x)| \in \Gamma$ for real. Another way to see it is to say that

$$(\Gamma \cup \{0\})^n = \coprod_{k=0}^{n} (\coprod_{\binom{n}{k}} \Gamma^k)).$$

So that we can always replace Γ_0 by Γ.
We have chosen to systematically omit this point in the proofs, in order to avoid some unprofitable complications. The careful reader should have this in mind.

4.1.4 Dimension for the Γ-sort

If $(\Gamma, <)$ is a totally ordered abelian group, the intervals $]a, b[:= \{\gamma \in \Gamma \mid a < \gamma < b\}$ form a basis of neighbourhood for some topology, which is called the *interval topology.* We will always refer to this topology on Γ.

Remind that if K is an algebraically closed non-trivially valued field, $\Gamma := |K^*|$ is a totally ordered divisible commutative group.
We will work with $\Gamma_0 := \Gamma \cup \{0\}$. On $\Gamma \cup \{0\}$, the group law is extended by $\gamma \cdot 0 = 0$, and $0 \leq \gamma$ for all $\gamma \in \Gamma$. On Γ_0, we consider the topology generated by the open intervals $]a, b[$ (with $a, b \in \Gamma$) and by $\{0\}$ (hence 0 is discrete in Γ_0).
A definable subset of $(\Gamma_0)^n$ is then a boolean combination of subsets

$$\{\gamma \in (\Gamma_0)^n \mid a\gamma^u \leq b\gamma^v\}$$

where $a, b \in \Gamma_0$ and $u, v \in \mathbb{N}^n$.

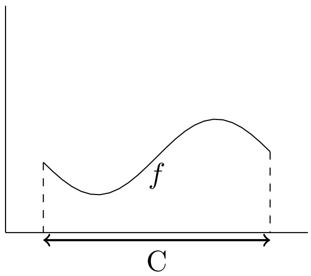
5. The case of totally ordered abelian divisible groups is only one case (and maybe the simplest) of an o-minimal theory, which is exposed in the excellent book [vdD98].

Definition 4.1.3. Let $n \in \mathbb{N}$, $(i_1, \ldots, i_n) \in \{0,1\}^n$. An $(i_1, \ldots, i_n)$**-cell** is a definable subset C of $(\Gamma_0)^n$ that we define inductively:

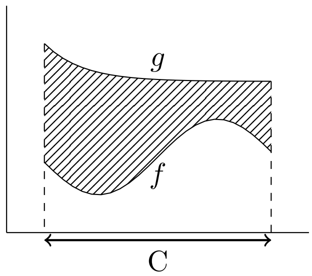
i) A (0)-cell is a singleton $\{\gamma\}$ with $\gamma \in \Gamma_0$.
A (1)-cell is in interval $]a, b[$ of Γ_0.

ii) If $C \subset \Gamma^n$ is a $(i_1, \ldots, i_n)$-cell, and $f, g : C \to \Gamma_0$ are continuous definable maps, and assume that for all $x \in C$, $f(x) < g(x)$. Then
$\mathrm{Graph}(f) := \{(x, y) \in (\Gamma_0)^n \times \Gamma_0 \mid x \in C \text{ and } y = f(x)\}$ is an $(i_1, \ldots, i_n, 0)$-cell.
$]g, f[:= \{(x, y) \in (\Gamma_0)^n \times \Gamma_0 \mid x \in C \text{ and } f(x) < y < g(x)\}$ is an $(i_1, \ldots, i_n, 1)$-cell.

A cell is an $(i_1, \ldots, i_n)$-cell for some $(i_1, \ldots, i_n)$, and a cell of dimension d is an $(i_1, \ldots, i_n)$-cell with $i_1 + \ldots + i_n = d$.



(a) Graph(f), an $(i_1, \ldots, i_n, 0)$-cell



(b) $]f, g[$, an $(i_1, \ldots, i_n, 1)$-cell

Figure 4.1: We suppose that $C \subset \Gamma^n$ is an $(i_1, \ldots, i_n)$-cell and that $f, g : C \to \Gamma$ are continuous definable functions such that $f < g$. The horizontal axis represents C, and the vertical axis represent Γ. On the left the graph of we obtain Graph(f). On the right, the hatched area represents $]f, g[$.

The following result can be found in [vdD98, 3.2.11]

Theorem 4.1.4 (Cell decomposition theorem)**.** *Let X be a definable subset of $(\Gamma_0)^n$. There exists a partition*

$$X = \coprod_{i=1}^{m} C_i$$

where each C_i is a cell.

Definition 4.1.5. Let X be a definable subset of $(\Gamma_0)^n$. The following numbers are equal, and will be denoted by $\dim(X)$.

1. $$\max_{\substack{C \subset X \\ C \text{ is a cell}}} (\dim(C))$$

2. $$\max_{k=1\ldots M} (\dim(C_k))$$
   where $X = \coprod_k C_k$ is any decomposition of X into cells. This number does not depend on the decomposition.

3. The greatest integer d such that there exists a coordinate projection $\pi : (\Gamma_0)^n \to (\Gamma_0)^d$ for which $\pi(X)$ has non empty interior.

These equivalence are proven in [vdD98, part 4], like this proposition:

Proposition 4.1.6. *[vdD98, 4.1.6] Let $f : S \to T$ be a definable maps between definable sets of some Γ_0^n, then* $\dim(f(S)) \leq \dim(S)$.

4.2 Mixed dimension of definable subsets of $K^m \times \Gamma^n$

In this part, we want to define what relative cells $C \subset (K^{alg})^m \times \Gamma^n$ are[6]. They will be the data of a definable subset $X \subset K^m$, $(i_1, \dots, i_n)$ a sequence of 0's and 1's, and C will be a nice family of $(i_1, \dots, i_n)$'s parametrized by X.

In this section, *definable* will have two possible meanings:

- Either, this will mean definable in ACVF the theory of algebraically closed field. In this context, K is an arbitrary model of ACVF, in particular, the valuation need not to be of height 1.
- Or definable will mean definable in the language $\mathcal{L}_{an}^D$, in other words, $S \subset (K^{alg})^m \times \Gamma^n$ is a subanalytic set (which according to the definition we have given means that in fact $S \subset ((K^{\text{alg}})^\circ)^m \times \Gamma^n)$. In this context, K is a complete non-Archimedean field, which in particular means that we have a fixed norm $|\cdot| : K \to \mathbb{R}_+$, hence an embedding of $(\Gamma, \cdot)$ in $(\mathbb{R}_+^*, \times)$.

However, we will give a uniform treatment of the cell decomposition, and of the mixed dimension that we will build for definable subsets of $(K^{alg})^m \times \Gamma^n$, which is rendered possible by the analogous properties of the dimension of subsets of K^n definable in ACVF or subanalytic. As for the definable subsets of Γ^n, they are the same in both theories (and have been described in section 4.1.4).

4.2.1 Mixed cell decomposition theorem

In the following definitions, we will write Γ_0 to insist that this is really what we will be looking at, but afterwards, we will switch to the notation Γ.

Definition 4.2.1. Let $X \subset (K^{alg})^m \times (\Gamma_0)^n$ be definable set. We will set:

$$\mathcal{C}^\circ(X) := \{f : X \to \Gamma_0 \mid f \text{ is continuous and definable}\} \cup \{+\infty\}.$$

Definition 4.2.2. Let $X \subset K^m$ be a nonempty definable subset (by convention, we consider that K^0 is a single point). For $(i_1, \dots, i_n) \in \{0,1\}^n$ we define inductively **X-$(i_1, \dots, i_n)$-cells**, which are definable subsets of $X \times (\Gamma_0)^n$:

i) if $m = 0$, i.e. $(i_1, \dots, i_n) = \emptyset$, then $X \subset K^m$ is the only $X - set$-cell.

ii) If $C \subset X \times (\Gamma_0)^n$ is an $X - (i_1, \dots, i_n)$-cell, and $f, g \in \mathcal{C}^\circ(X)$, such that for all $x \in C$, $f(x) < g(x)$, then

$$\text{Graph}(f) := \{(x, \gamma) \in C \times \Gamma_0 \mid \gamma = f(x)\} \text{ is an } X - (i_1, \dots, i_n, 0)\text{-cell.}$$
$$]f, g[:= \{(x, \gamma) \in C \times \Gamma_0 \mid f(x) < \gamma < g(x)\} \text{ is an } X - (i_1, \dots, i_n, 1)\text{-cell.}$$

Remark 4.2.3. If $X \subset K^m$ is a definable set, C is an $X - (i_1, \dots, i_n)$-cell, then for all $x \in X$, if we set

$$C_x := \{\gamma \in \Gamma^n \mid (x, \gamma) \in C\}$$

then $C_x \subset \Gamma^n$ is an $(i_1, \dots, i_n)$-cell in the sense of the definition 4.1.3, and one has to think of C as a definable continuous family of $(i_1, \dots, i_n)$-cells parametrized by X. Remark that if $(i_1, \dots, i_n) \neq (j_1, \dots, j_n)$, then a $X - (i_1, \dots, i_n)$-cell and a $X - (j_1, \dots, j_n)$-cell are necessarily different. This is true because anyway a $(i_1, \dots, i_n)$-cell and a $(j_1, \dots, j_n)$-cell are different. One way to see this is to remark that the projection of an $(i_1, \dots, i_n)$-cell along the first k coordinates is an $(i_1, \dots, i_k)$-cell.

6. We should say $K^m \times (\Gamma_0)^n$ but following the remark made at the end of the introduction, we will ignore 0.

Definition 4.2.4. Let $X \subset K^m$ be definable and $C \subset X \times \Gamma^n$ a $X-(i_1,\dots,i_n)$-cell. Let $d_1 = \dim(X)$ and $d_2 = i_1+\ldots+i_n$. We say that C is a cell of dimension $(d_1,d_2) \in \mathbb{N}^2$.

We make this first observation:

Proposition 4.2.5. *Let $C \subset (K^{alg})^m \times \Gamma^n$ be a cell. Then*

$$\dim(C) = (m,n) \Leftrightarrow C \text{ has non-empty interior} \Leftrightarrow C \text{ is somewhere dense.}$$
$$\dim(C) \neq (m,n) \Leftrightarrow C \text{ has empty interior} \Leftrightarrow C \text{ is nowhere dense.}$$

Proof. We first prove that $\Big(\dim(C) = (m,n)\Big) \Rightarrow C$ has non-empty interior:
in that case C is an $X-(1,\dots,1)$-cell with $\dim(X)=m$. Then according to proposition 3.1.16 (2), X has non-empty interior, hence we can assume that X is open. Then, using the definition of a cell, one proves by induction on n that an $X-(1,\dots,1)$-cell is open.

Now $\Big(C$ has non-empty interior $\Big) \Rightarrow C$ is somewhere dense is clear.

Eventually, we prove that $\Big(\dim(C) \neq (m,n)\Big) \Rightarrow C$ is nowhere dense.
If C is an $X-(i_1,\dots,i_n)$-cell, then according to proposition 3.1.16 (iii), if $\dim(X) < m$, X is nowhere dense, hence $X \times \Gamma^n$ also. Hence the result follows in that case. Let us then assume that $\dim(X) = m$ (hence, since $\dim(C) \neq (m,n)$, for some k, $i_k = 0$). In fact using 3.1.30, 3.1.24 and the preceding argument, we can even assume that X is open. If C was dense at a point $(x,\gamma) \in (K^{alg})^m \times \Gamma^n$, we could assume that $(x,\gamma) \in C$. But, it is easy to see that when C is an $X-(i_1,\dots,i_n)$-cell with X open, then locally around a point of C, $\overline{C} = C$ (in other words, we have the equality of germs $(\overline{C},c) = (C,c)$), and if $i_k = 0$ for some k, then C has no interior points.

□

Proposition 4.2.6. *Let $S \subset (K^{alg})^m \times \Gamma^n \times \Gamma$ be a subanalytic set. Assume that for all $z \in (K^{alg})^m \times \Gamma^n$ the fibre $S_z := \{(\gamma \in \Gamma \mid (z,\gamma) \in S\}$ is finite. Then there exists an integer N such that for all $z \in (K^{alg})^m \times \Gamma^n$, Card$(S_z) \leq N$.*

Proof. Following the proof of proposition 3.1.36, we can assume that S is defined by a formula φ involving only the variables $(x,\alpha) \in (K^{alg})^m \times \Gamma^n$ and a conjunction of formulas

$$\{(x,\alpha,\gamma) \in \left((K^{\text{alg}})^\circ\right)^m \times \Gamma^n \times \Gamma \mid |f_i(x)|\alpha^{u_i} = \gamma^{a_i}\} \ i = 1\ldots N$$

with $f_i : (K^{alg})^m \to K^{alg}$ a definable function, $u_i \in \mathbb{Z}^n$, and $a_i \in \mathbb{N}^*$.
Hence, if $z = (x,\alpha) \in (K^{alg})^m \times \Gamma^n$,

$$S_z \subset \left\{ \sqrt[a_i]{|f_i(x)|\alpha^{u_i}} \right\}_{i=1\ldots N}$$

hence Card$(S_z) \leq N$. □

We now give the following lemma which is the essential idea of the proof of the cell decomposition theorem, and which is a complete adaptation of the proof [vdD98, th 4.2.11]:

Fact 4.2.7 (The essence of o-minimality)**.** Let $A \subset \Gamma$ be a definable subset.

1. There exists $M, N \in \mathbb{N}$, two sequences

$$a_1 < b_1 \leq a_2 < b_2 \leq a_3 < \ldots b_{N-1} \leq a_N < b_N$$

and $c_1 < c_2 < \ldots < c_M$ of elements of Γ (we allow $a_1 = -\infty$ and $b_N = +\infty$), and some $\lceil_i, \rfloor_i \in \{], [\}$ for $i = 1 \ldots N$ such that

$$A = (\bigcup_{i=1}^{M} \lceil_i a_i, b_i \rfloor_i) \cup (\bigcup_{j=1}^{N} \{c_j\})$$

$$\text{for } i = 1 \ldots N,\ j = 1 \ldots M,\ c_j \notin \{a_i, b_i\}$$

$$\text{We can not have} \rfloor_i =] \text{ and} \lceil_{i+1} = [.$$

This definition forces the sets $\lceil_i a_i, b_i \rfloor_i$ and $\{c_j\}$ to be the maximal intervals contained in A, hence are uniquely determined by A, in particular the couple (M, N) is determined by A, and we will say that A is of type (M, N).

2. $\mathrm{Bd}(A) = \{a_i, b_i, c_j\ i = 1 \ldots N,\ j = 1 \ldots M\}$ where $\mathrm{Bd}(A) = \overline{A} \setminus A$.
3. If $S \subset (K^{alg})^m \times \Gamma^n \times \Gamma$ is a definable set, for $(M, N) \in \mathbb{N}^2$ let

$$S^{(M,N)} = \{z \in (K^{alg})^m \times \Gamma^n \mid S_z \text{ is a definable set of } \Gamma \text{ of type } (M, N)\}.$$

Then $S^{(M,N)}$ is definable.
4. Let $S \subset (K^{alg})^m \times \Gamma^n \times \Gamma$ be a definable set and $(M, N) \in \mathbb{N}^2$. If $z \in S^{(M,N)}$, $S_z \subset \Gamma$ is by definition a definable set of Γ of type (M, N), and has a canonical decomposition

$$S_z = \Big(\bigcup_{i=1}^{M} \lceil_i a_i(z), b_i(z) \rfloor_i\Big) \cup \Big(\bigcup_{j=1}^{N} \{c_j(z)\}\Big)$$

as in (1). Then the functions $a_i, b_i, c_j : S^{(M,N)} \to \Gamma$ are definable.
5. Let $S \subset (K^{alg})^m \times \Gamma^n \times \Gamma$ be a subanalytic set. There is only a finite number of $(M, N) \in \mathbb{N}^2$ such that $S^{(M,N)} \neq \emptyset$. In other words, only finitely many types are realized by $\{S_z\},\ z \in (K^{alg})^m \times \Gamma$, the family of definable sets of Γ.

The proof of our mixed cell decomposition theorem can now be done as in [vdD98, th 4.2.11]:

Theorem 4.2.8. *Let $S \subset (K^{alg})^m \times \Gamma^n$ be a definable set. There exists a partition*

$$S = \coprod_{i=1}^{l} C_i$$

where each C_i is a cell of $(K^{alg})^m \times \Gamma^n$.

Proof. We prove this by induction on n.

For $n = 0$, $S \subset \left((K^{\mathrm{alg}})^\circ\right)^m$ is just a definable subset of $\left((K^{\mathrm{alg}})^\circ\right)^m$, hence is a $S - \emptyset - cell$ by definition.

Let $n \geq 0$, and $S \subset (K^{alg})^m \times \Gamma^n \times \Gamma$ be a definable set. According to lemma 4.2.7, there exists only a finite number of possible couples $(M, N) \in \mathbb{N}^2$, say $\mathcal{Q}$, such that $S^{(M,N)} \neq \emptyset$, i.e. such that for some $z \in (K^{alg})^m \times \Gamma^n$, S_z has type (M, N). For such a couple (M, N), we can then define functions $a_i, b_i, c_j : S^{(M,N)} \to \Gamma$ which are definable [7] (see lemma 4.2.7 (4)). We then define

$$]a_i^{(M,N)}, b_i^{(M,N)}[:= \{(z, \gamma) \mid z \in S^{(M,N)} \text{ and } a_i(z) < \gamma < b_i(z)\}$$

7. According to definition of the a_i and $b_i's$ we've given in lemma 4.2.7, it is possible that $a_1 = 0$ and $b_N = +\infty$. This is then not a problem, and what follows is correct for this cases too. One way to give a precise solution is to treat separately the cases where $a_1 = 0$ and $b_N = +\infty$, and to remark that the sets $\{z \in S^{(M,N)} \mid a_1(z) = 0\}$ and $\{z \in S^{(M,N)} \mid b_N(z) = +\infty\}$ are definable.

$$\{c_j^{(M,N)}\} := \Big\{(z,\gamma) \mid z \in S^{(M,N)} \text{ and } c_j(z) = \gamma\Big\}.$$

By construction, S is the disjoint union

$$S = \bigcup_{(M,N)\in\mathcal{Q}} \Big(\bigcup_{i=1}^{M}]a_i^{(M,N)}, b_i^{(M,N)}[\Big) \cup \Big(\bigcup_{j=1}^{N} \{c_j^{M,N}\}\Big).$$

Now, according to proposition 3.1.36, we can shrink the definable sets $S^{(M,N)}$ and assume that the $a_i, b_i, c_j's$ are continuous. In addition, by induction hypothesis, decomposing each $S^{(M,N)}$ if necessary, we can assume that $S^{(M,N)}$ is an $X - (i_1, \ldots, i_n)$-cell where $X \subset (K^{alg})^m$ is a definable set and $(i_1, \ldots, i_n) \in \{0,1\}^n$. In that case, $]a_i^{(M,N)}, b_i^{(M,N)}[$ is a $X - (i_1, \ldots, i_n, 1)$-cell and $\{c_j^{(M,N)}\}$ is a $X - (i_1, \ldots, i_n, 0)$-cell. □

4.2.2 Mixed dimension

The mixed cell decomposition theorem will allow us to develop a theory of dimension for definable subsets of $(K^{alg})^m \times \Gamma^n$ which extends the dimension theories for definable subsets of $(K^{alg})^m$ (as exposed in section 3.1 and 3.2) and for definable subsets of Γ^n (as presented in section 4.1.4).
Naively, if $S \subset (K^{alg})^m \times \Gamma^n$ is a definable subset, we would like $\dim(S)$ to be equal to $(d_1, d_2) \in \mathbb{N}^2$ where d_1 would be some K-dimension and d_2 some Γ-dimension.

Example 4.2.9. If $S \subset K^{alg} \times \Gamma^2$ is defined by

$$S = S_1 \bigcup S_2 = \{(x,1,1) \mid x \in K\} \bigcup \{(0,\gamma_1,\gamma_2) \mid (\gamma_1,\gamma_2) \in \Gamma^2\}$$

the situation is quite embarrassing. Clearly, we would like that $\dim(S_1) = (1,0)$ and $\dim(S_2) = (0,2)$. Note by the way that S_1 is a $K - (0,0)$-cell, hence a cell of dimension $(1,0)$ according to definition 4.2.4, and S_2 is a $\{0\} - (1,1)$-cell, hence a cell of dimension $(0,2)$. However, we do not want to choose between $(1,0)$ and $(0,2)$. Since $2 \geq 1$, we do not want to remove $(0,2)$. On the other hand, we do not want to remove $(1,0)$ because it tells us that inside S, S_1 has K-dimension 1, and as we will see in 4.2.12, since $S_2 \simeq \Gamma^2$, this K-dimension 1 is somehow stronger than the Γ dimension 2. The solution would be to keep both $(1,0)$ and $(0,2)$, which the following definition does.

Definition 4.2.10. Let $S \subset (K^{alg})^m \times \Gamma^n$ be a definable set. We define the dimension of S, denoted by $\dim(S)$, as the finite subset of $\mathbb{N}^2$:

$$\dim(S) = \{(d_1,d_2) \in \mathbb{N}^2 \mid S \text{ contains a cell C of dimension } (d_1,d_2)\}$$

where the dimension of a cell is the one given by definition 4.2.4.

Remark 4.2.11. If D is a finite subset of $\mathbb{N}^2$, we will represent it with one symbol • for each point of D. For instance to the subset

$$D_1 = \{(0,0),(0,3),(0,4),(1,4),(2,0),(2,1),(2,2),(4,0),(4,1)\}$$

will will associate the diagram

1. We will need to compare dimensions, hence we will consider $\mathbb{N}^2$ equipped with the partial order $\leq$ defined by $(d_1,d_2) \leq (d_1', d_2')$ if $d_1 \leq d_1'$ and $d_2 \leq d_2'$.

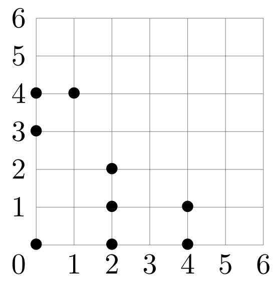


Figure 4.2: Diagram associated to D_1.

2. We will say that a set $D \subset \mathbb{N}^2$ is a lower set if for all $x \in \mathbb{N}^2, y \in D$, if $x \leq y$, then $x \in D$. In other words, if one represents D in the plane, it is a lower set if it is stable in the directions $\leftarrow$ and $\downarrow$. For instance D_1 is not a lower set. If we denote by D_2 the smallest lower set which contains D_1, then D_2 is represented by the diagram

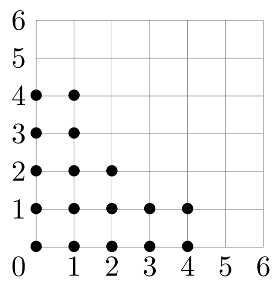


Figure 4.3: The diagram of D_2, the smallest lower set containing D_1.

A finite lower set of $\mathbb{N}^2$ marks out a bounded convex subset of $\mathbb{R}_+^2$ that we fill in with some small dots. With this graphical convention D_2 is represented by the diagram

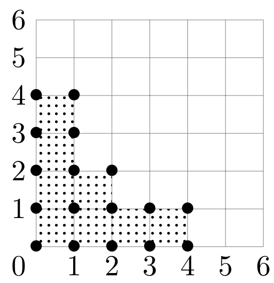
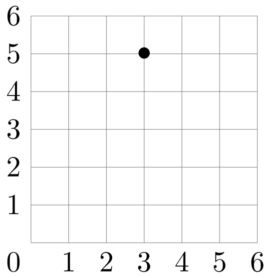


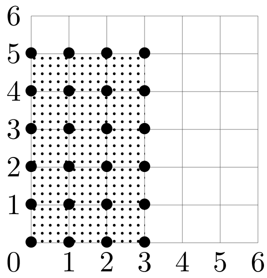
Figure 4.4: The diagram of D_2 which has been filled.

3. If A and B are subsets of $\mathbb{N}^2$, $A \leq B$ will mean $A \subset B$.
4. If S is a definable set of $(K^{alg})^m \times \Gamma^n$, $\dim(S)$ is a finite lower set of $\mathbb{N}^2$. Indeed, if C is a $X - (i_1, \ldots, i_n)$-cell in S with $d_1 = \dim(X)$ and $d_2 = i_1 + \ldots + i_n$, and if $(d_1', d_2') \leq (d_1, d_2)$, we can choose $Y \subset X$ such that $\dim(Y) = d_1'$ and $C' := C \cap (Y \times \Gamma^n)$ is a $Y - (i_1, \ldots, i_n)$-cell of dimension (d_1', d_2). It is now easy to find a cell $C'' \subset C'$ of dimension (d_1', d_2'). We only have to decrease the Γ-dimension. For instance, if A is $X - (i_1, \ldots, i_n)$-cell, and $B =]f, g[$ a $X - (i_1, \ldots, i_n, 1)$-cell above A, then Graph$(\sqrt{fg})$ is a $X - (i_1, \ldots, i_n, 0)$-cell included in B.

5. We will have to add dimensions. If A and B are subsets of $\mathbb{N}^2$, we will set
$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$
More generally, if $A \subset \mathbb{N}^2$ and $B \subset \mathbb{Z}^2$, we will set
$$A + B = \{a + b \in \mathbb{N}^2 \mid a \in A, b \in B\}.$$
6. We will have to take maximum of dimensions. If A, B are subsets of $\mathbb{N}^2$, we will set
$$\max(A, B) = A \cup B.$$
Note that if A and B are lower sets, $A \cup B$ is also a lower set.
7. Eventually, if $(d_1, d_2) \in \mathbb{N}^2$, we will set
$$\langle (d_1, d_2) \rangle = \{x \in \mathbb{N}^2 \mid x \leq (d_1, d_2)\}.$$
This is the smallest lower set which contains (d_1, d_2), and (d_1, d_2) is the greatest element (w.r.t. $\leq$) of $\langle (d_1, d_2) \rangle$. We draw a picture to describe the situation if $(d_1, d_2) = (3, 5)$:



(a) Diagram of $\{(3, 5)\}$



(b) $D_3 = \langle (3, 5) \rangle$

Figure 4.5: On the left the diagram of $\{3, 5\}$. On the right, $D_3 := \langle (3, 5) \rangle$, the smallest lower set containing $(3, 5)$. For instance, if $S = \left((K^{\mathrm{alg}})^\circ\right)^3 \times \Gamma^5$, then $\dim(S) = D_3$.

At this point we want to explain that there is no ambiguity in the meaning of the dimension of C, if C is a $X - (i_1, \ldots, i_n)$-cell. According to definition 4.2.4, we have defined $\dim(C)$ as a couple (d_1, d_2) where $d_1 = \dim(X)$ and $d_2 = i_1 + \ldots + i_n$. Now, according to definition 4.2.10, we have also defined $\dim(X)$ as a subset $A \subset \mathbb{N}^2$. The compatibility that we expect (and which will be proved in proposition 4.2.14(2)) is that $\langle \dim(C) \rangle = \dim(C)$ where on the left side we took the dimension of a cell as defined in 4.2.4, and on the right side, we consider the definition of the definable set C as defined in 4.2.10. Hence, definitions 4.2.4 and 4.2.10 are compatible.

Lemma 4.2.12. *Let $P \subset \Gamma^n$ be definable, and $f : P \to (K^{alg})^m$ some definable function. Then f has finite image.*

Proof. One can assume that $m = 1$. Hence, if $f(P)$ was not finite, it would contain a ball (thanks to the description of definable sets of K^{alg} as Swizz cheese). This would imply that $\mathrm{Card}(\Gamma) \geq \mathrm{Card}(P) \geq \mathrm{Card}(f(P)) \geq \mathrm{Card}(K)$ which is false for some extensions of K^{alg}. □

Lemma 4.2.13.

1. *Let $C \subset (K^{alg})^m \times \Gamma^n$ be a cell of dimension $d = (d_1, d_2)$, and assume that*
$$C = C_1 \cup \ldots \cup C_N$$
where each C_i is a cell. Then
$$d = \max_{j=1\ldots N} \dim(C_j).$$

2. *If $f : C \hookrightarrow C'$ is an injective definable map between cells,* $\dim(C) \leq \dim(C')$.

Proof.

1. Let C be a $X - (i_1, \ldots, i_n)$-cell and $\{C_j\}_{j=1\ldots N}$ some $X_j - (i_1^j, \ldots, i_n^j)$-cells such that $C = \cup_{j=1\ldots N} C_j$. Then for all j, it is easy to see that $X_j \subset X$, hence that $\dim(X_j) \leq \dim(X)$.
In addition if $x \in C_j$, then $(C_j)_x := \{\gamma \in \Gamma^n \mid (x, \gamma) \in C_j\}$ is a Γ-definable set included in C_x, this proves that $\dim(C_j) \leq d$.
Now, $\max_{j=1\ldots N}(\dim(X_j)) = \dim(X)$ (see 3.1.18). Let then
$$X' := X \setminus (\bigcup_{\substack{i \text{ such that} \\ \dim(X_i) < \dim(X)}} X_i).$$
Then $\dim(X') = \dim(X)$ and replacing X by X', we can assume that $\dim(X_j) = \dim(X)$ for all j.
Hence if $x \in X$, $C_x = \cup_j (C_j)_x$, hence for some j, $(C_j)_x$ has the same dimension as C_x. So C_j is a cell of the same dimension as C.
2. Let C be a $X - (i_1, \ldots, i_l)$-cell and C' a $Y - (j_1, \ldots, j_n)$-cell and $f : C \hookrightarrow C'$ a definable injective map. We first remark [8] that the first projection $\pi : C \to X$ has a definable section $s : X \hookrightarrow C$. We then obtain an injective map $g : X \hookrightarrow C'$. According to propositions 3.1.34, 3.1.36, we can write $X = X_1 \cup \ldots \cup X_N$ such that each $g_{|X_i}$ is continuous, and since one of the $X_i's$ must have the same dimension as X, we can assume that $g : X \to Y \times \Gamma^n$ is continuous [9]. For $i = 1 \ldots n$, let g_i be the composite of g with the i-th projection on Γ. According to 3.1.36, we can even assume that $g_i = \sqrt[a_i]{|G_i|}$ for some continuous definable function $G_i : X \to (K^{alg})^*$. Let then $x \in X$ such that $\dim_x(X) = \dim(X)$ (see the previous section). Then there exists a definable neighbourhood V of x such that the $|G_i|$'s are constant on V; moreover $\dim(V) = \dim(X)$. Hence if we project on Y we now obtain a definable

8. This can be proved by induction on l. For instance if $C =]f, g[$ is a $X - (1)$-cell, we take
$$\begin{array}{cccc} s: & X & \hookrightarrow & C \\ & x & \mapsto & (x, \frac{f(x)+g(x)}{2}) \end{array}.$$

9. Remind that we are in fact looking at maps $g : X \to Y \times \{\Gamma \cup \{0\}\}^m$. But since
$$(\Gamma \cup \{0\})^n = \coprod_{k=0}^{n} (\coprod_{\binom{n}{k}} \Gamma^k))$$
we can stratify X such that g is actually a map $g : X \to Y \times \Gamma^k$ for some $k \leq n$.

injection $V \hookrightarrow Y$. Hence $\dim(Y) \geq \dim(V) = \dim(X)$. This proves that $d_1 \leq d'_1$. Now, let $x \in X$. Then f induces a map $f : C_x \to C' \subset Y \times \Gamma^n$ which is not necessarily injective. However, if we project on Y, we obtain a map $\alpha : C_x \to Y$ with $C_x \subset \Gamma^l$, and $Y \subset (K^{alg})^p$ for some p. Hence according to lemma 4.2.12, $\alpha(C_x)$ is finite, say $\{y_1, \ldots, y_M\}$. Hence there exists a definable subset $P \subset C_x$ such that $\alpha_{|P} : P \to Y$ is constant, with image y say, and such that $\dim(P) = \dim(C_x) = d_2$. Now, $f_{|P} : P \to C'$ is injective and even induces an injective map $P \hookrightarrow (C')_y$ which is a Γ-definable set of dimension d'_2. Hence $d_2 = \dim(P) \leq d'_2$.

□

We now give some general properties satisfied by dimension of definable subsets of $(K^{alg})^m \times \Gamma^n$.

Proposition 4.2.14.

1. *Let* $X \subset Y \subset (K^{alg})^m \times \Gamma^n$ *be definable subsets. Then* $\dim(X) \leq \dim(Y)$.
2. *Let* $C \subset (K^{alg})^m \times \Gamma^n$ *be a cell of dimension* (d_1, d_2) *(according to definition 4.2.4). Then* $\dim(C) = \langle (d_1, d_2) \rangle$ *(according to definition 4.2.10).*
3. *If* $f : X \hookrightarrow Y$ *is a definable injection of definable subsets of* $(K^{alg})^m \times \Gamma^n$ *(resp.* $(K^{alg})^{m'} \times \Gamma^{n'}$*), then* $\dim(X) \leq \dim(Y)$.
   If f *is bijective,* $\dim(X) = \dim(Y)$.
4. $\dim(X \cup Y) = \max(\dim(X), \dim(Y))$.
5. *If* $X = \cup_{i=1}^N C_i$ *is a union of cells of dimension* (d_1^i, d_2^i) *(according to definition 4.2.4), then* $\dim(X) = \max_i \langle (d_1^i, d_2^i) \rangle$.

Proof.

1. Is clear.
2. Is a consequence of lemma 4.2.13 (2).
3. For the first assertion, let $C \subset X$ be a cell of dimension (d_1, d_2). Then according to the cell decomposition theorem, we can find a decomposition $f(C) = C'_1 \cup \ldots \cup C'_N$ in N cells. Let then $C_i := f^{-1}(C'_i)$. Then $C = C_1 \cup \ldots \cup C_N$ hence according to lemma 4.2.13 (1), for some i, $\dim(C_i) = \dim(C)$. Since $f_{|C_i} : C_i \hookrightarrow C'_i$ is injective, according to lemma 4.2.13 (1), $\dim(C_i) \leq \dim(C'_i) \leq \dim(Y)$. Hence this proves that $\dim(X) \leq \dim(Y)$.
   To prove the second assertion, we apply the first one to f and f^{-1}.
4. Clearly, $\max(\dim(X), \dim(Y)) \leq \dim(X \cup Y)$. Conversely, let $C \subset X \cup Y$ be a cell. We apply the cell decomposition theorem to $X \cap C$ and $Y \cap C$:
   $$X \cap C = C_1 \cup \ldots \cup C_M, \text{ and } Y \cap C = C'_1 \cup \ldots \cup C'_N.$$
   Then, $C = C_1 \cup \ldots \cup C_M \cup C'_1 \cup \ldots \cup C'_N$, hence according to lemma 4.2.13 (2), for some i $\dim(C_i) = \dim(C)$ or for some j, $\dim(C'_j) = \dim(C)$, hence $\dim(C) \leq \max(\dim(X), \dim(Y))$.
5. Is a consequence of (2) and (4).

□

4.2.3 After you project

Lemma 4.2.15. *Let us assume that* K *is algebraically closed.*
Let $f_1, \ldots, f_N \in K[T]$. *There exists a decomposition* $K = V_1 \cup \ldots \cup V_M$ *of* K *in definable subsets*[10] *and for each* $j = 1 \ldots m$, *a point* $a_j \in K$, *some integers* $n_{i,j} \in \mathbb{N}$ *and some*

10. which are in fact semialgebraic according to [LR96, 4.7].

$c_{i,j} \in \Gamma_0$ *such that*

$$\text{for all } j = 1 \ldots M, i = 1 \ldots N, x \in V_j, \quad |f_i(x)| = c_{i,j}|x - a_j|^{n_{i,j}}.$$

Proof. First, it suffices to treat the case when the functions f_i have degree 1, and even when $f_i(T) = T - \alpha_i$ for some $\alpha_i \in K$. Then one considers the family $\{V_1, \ldots, V_M\}$ of definable sets of K which are boolean combination of the closed and open balls

$$B(\alpha_i, |\alpha_i - \alpha_j|)_{1 \le i,j \le N}.$$

It is a simple calculation to see that this works for this choice of V_j's.
One way to see it (in the case where K is complete), is that the norm of a polynomial map f_i on $\mathbb{A}_K^{1,an}$ is piecewise linear along a tree T_i on $\mathbb{A}_K^{1,an}$. One then consider the union of the $T_i's$ which is still a tree, and then one restrict to the segments of this tree to carry the proof. □

Lemma 4.2.16. *Let $C \subset K^{alg} \times \Gamma^n$ be a cell of dimension $(1, d)$.*
Let $\pi : (K^{alg}) \times \Gamma^n \to \Gamma^n$ be the second projection.
Then

$$\dim(\pi(C)) \le d + 1.$$

Proof. Assume that C is a $X - ((i_1, \ldots, i_n))$-cell, where X is a definable subset of K with $\dim(X) = 1$, and $i_1 + \ldots + i_n = d$. Then by definition of a cell, there exists a sequence $C_0, \ldots C_n$ such that

- $C_0 = X$.
- For each $k = 0 \ldots n$, C_k is a $X - (i_1, \ldots, i_k)$-cell.
- For each $k = 0 \ldots n - 1$
  - if $i_{k+1} = 0$, there exists a (continuous) definable map $f_k : C_k \to \Gamma$ such that $C_{k+1} = \mathrm{Graph}(f_k)$.
  - If $i_{k+1} = 1$, there exist (continuous) definable maps $f_k, g_k : C_k \to \Gamma$ such that $f_k < g_k$ and $C_{k+1} =]f_k, g_k[$.

Remind that by construction, C_k is a definable subset of $X \times \Gamma^k \subset K^{alg} \times \Gamma^k$, hence $f_k, g_k : X \times \Gamma^k \to \Gamma$ are definable maps. According to proposition 3.1.36, we can shrink X such that

$$\begin{array}{rccc} f_k: & X \times \Gamma^k & \to & \Gamma \\ & (x, \gamma) & \mapsto & \sqrt[d_k]{|F_k(x)|\gamma^{u_k}} \end{array}$$

for some definable $F_k : X \to K^{alg}$, some $u_k \in \mathbb{Z}^k, d_k \in \mathbb{N}^*$, and similarly,

$$g_k(x, \gamma) = \sqrt[e_k]{|G_k(x)|\gamma^{v_k}}.$$

Now, according to [LR96, 3.3], we can again shrink X and assume that $F_k, G_k \in K(T)$ are fractions without pole on X.
Now, according to lemma 4.2.15, we can again shrink X and assume that
there exist some $a \in K$, $m_k, n_k \in \mathbb{Z}, b_k, c_k \in \Gamma$ such that for all $x \in X$

- $|F_k(x)| = b_k|x - a|^{m_k}$
- $|G_k(x)| = c_k|x - a|^{n_k}$.

We can again shrink X so that $I = \{|x - a| \mid x \in X\}$ is a cell of Γ (that is to say, a singleton or an open interval). Then if we consider the map

$$\begin{array}{rccc} \varphi: & X \times \Gamma^n & \to & \Gamma \times \Gamma^n \\ & (x, \gamma) & \mapsto & (|x - a|, \gamma) \end{array}$$

it is easily checked that $C' := \varphi(C) \subset \Gamma \times \Gamma^n$ is a cell:

– A $(1, i_1, \ldots, i_n)$-cell if I is an interval, hence of dimension $d+1$.
– A $(0, i_1, \ldots, i_n)$-cell if I is a singleton, hence of dimension d.

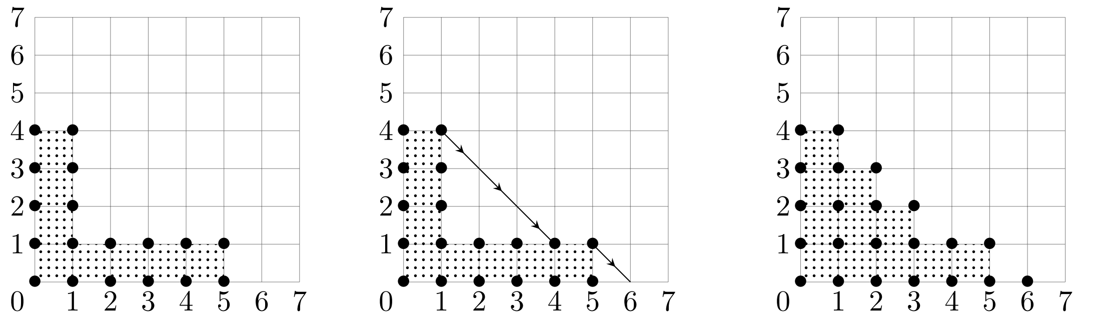
Now if $p : \Gamma \times \Gamma^n \to \Gamma^n$ is the coordinate projection along the last n coordinates, by construction, $\pi = p \circ \varphi$. Hence $\pi(C) = p(\varphi(C)) = p(C')$ and since the possible dimensions of C' are $d, d+1$, the possible dimensions for $\pi(C)$ are also $d, d+1$ (see 4.1.6). □

Let $\mathcal{P} = \{(-d, d) \mid d \in \mathbb{N}\}$.

Proposition 4.2.17. *Let $f : X \to Y$ be a definable map of definable subsets, $X \subset (K^{alg})^m \times \Gamma^n$, $Y \subset (K^{alg})^{m'} \times \Gamma^{n'}$. Then*

$$\dim(f(X)) \leq \max_{d \geq 0}(\dim(X) + (-d, d)) = \dim(X) + \mathcal{P}.$$

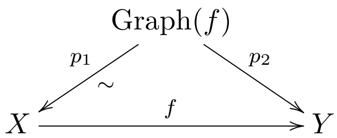
Before giving the proof, let us explain what this mean on a diagram. If $\dim(X) = D_2$ for instance, then the possibilities for $\dim(f(X))$ are encoded by the digram obtain by the digram of D_2 that we extend by adding new points, following arrows ↘ in the south-east direction.



(a) Diagram of D_4 (b) in between (c) $D_5 = \max_{d \geq 0}(D_4 + (-d, d))$

Figure 4.6: On the left, an example of a finite lower set D_4. For instance, if $S = (K^{\text{alg}})^\circ \times \Gamma^4 \cup \left((K^{\text{alg}})^\circ\right)^5 \times \Gamma \subset \left((K^{\text{alg}})^\circ\right)^6 \times \Gamma^4$, then $\dim(S) = D_4$. In the middle, we explain how we can obtain new dimensions for $\dim(f(X))$. On the right we obtain the finite lower set $D_5 = \max_{d \geq 0}(D_4 + (-d, d))$. The above proposition asserts that $\dim(f(X)) \leq D_5$.

Proof. Consider



where Graph(f) is a definable set of $(K^{alg})^{m+m'} \times \Gamma^{n+n'}$.

According to proposition 4.2.14 (3), $\dim(X) = \dim(\text{Graph}(f))$. Hence we can assume that $f = p_2$, that is to say, we can assume that f is a coordinate projection.

We can even assume that X is a cell, say a cell C of dimension (d_1, d_2). Let us assume that C is a $T - (i_1, \ldots, i_n)$-cell.

Making an induction, we can even assume that f is a projection along a single coordinate [11].

11. Somehow because $\mathcal{P} + \mathcal{P} = \mathcal{P}$. In other words, it is easy to prove that if the statement we want to prove is true for $f : X \to Y$ and $g : Y \to Z$, then it is also true for $g \circ f : X \to Z$.

We then distinguish the two cases corresponding to a projection along a variable of K^{alg} or of Γ [12]. **case K**. If f is the projection

$$f : (K^{alg})^m \times K^{alg} \times \Gamma^n \to (K^{alg})^m \times \Gamma^n$$

let $S := \pi(T)$ where

$$\pi : (K^{alg})^m \times K^{alg} \to (K^{alg})^m$$

is the projection that we imagine. Then after a partition of S, we can assume that $S = T(0)$ or $S = T(1)$ with respect to the projection π (see 3.1.19), and we treat these two cases separately.

If $S = T(0)$. Let then $D \subset f(C)$ be a cell, say an $U - (j_1, \ldots, j_n)$-cell. Then by definition $U \subset S$ hence $\dim(U) \leq \dim(T)$, and for $u \in U$, $\pi^{-1}(u)$ is finite. This implies that D_u is the projection along f of a finite number of $(i_1, \ldots, i_n)$-cells. Hence $\dim(D_u) \leq i_1 + \ldots + i_n$, but D_u is a $(j_1, \ldots, j_n)$-cell by hypothesis. This proves that $j_1 + \ldots + j_n \leq i_1 + \ldots + i_n$. Hence $\dim(D) \leq \dim(C)$.

If $S = T(1)$. In that case, according to proposition 3.1.19,

$$\dim(S) = \dim(T) - 1.$$

Since $S = \pi(T)$ by definition, it follows that $f(C) \subset S \times \Gamma^n$.

Let then $D \subset \pi(C)$ be cell. Say D is a $U - (j_1, \ldots, j_n)$-cell. If we prove that $\dim(D) \leq \max(\dim(C), \dim(C) + (-1, 1))$, this will conclude the proof. Necessarily, $U \subset S$, hence

$$\dim(U) \leq \dim(S) \leq d_1 - 1.$$

Now, if $u \in U$, then

$$D_u := \{\gamma \in \Gamma^n \mid (u, \gamma) \in D\}$$

is a $(j_1, \ldots, j_n)$-cell and $D_u \subset p(C_u)$ where

$$p : K \times \Gamma^n \to \Gamma^n$$

and

$$C_u := \{(x, \gamma) \in K \times \Gamma^n \mid (u, x, \gamma) \in C\}.$$

Now C_u can be decomposed in some cells, whose dimension are $(0, d)$ or $(1, d)$ with $d \leq d_2$.

Now the projection of a $(0, d)$-cell along p is clearly a d-cell of Γ^n.

According to lemma 4.2.16, the projection of a cell of dimension $(1, d)$ along p is a cell of Γ^n of dimension $\leq d + 1$. Hence, $\dim(D_u) \leq d + 1 \leq d_2 + 1$, which means that D is a cell of dimension

$$\dim(D) \leq (d_1 - 1, d_2 + 1) \leq \max(\dim(X), \dim(X) + (-1, 1)).$$

This ends the case of a projection along K^{alg}.

case Γ. If f is a projection

$$f : (K^{alg})^m \times \Gamma^n \times \Gamma \to (K^{alg})^m \times \Gamma^n$$

then it is clear that if $C \subset (K^{alg})^m \times \Gamma^{n+1}$, is a $T - (i_1, \ldots, i_n, i_{n+1})$-cell, then $f(C)$ is a $T - (i_1, \ldots, i_n)$-cell. In that case, we can even say that $\dim(f(C)) \leq \dim(C)$ (with a strict inequality if and only if $i_{n+1} = 1$). □

12. That case being much simple.

This was the last point needed to prove the theorem mentioned in the introduction:

Theorem 4.2.18. *The mixed dimension for definable sets of $(K^{alg})^m \times \Gamma^n$ satisfies the following properties:*

(i) *If $S \subset \Gamma^n$, then* $\dim(S) = \langle(0,d)\rangle$ *where d is the classical dimension of S as a subset of Γ^n as defined in section 4.1.4.*

(ii) *If $S \subset (K^{alg})^m$ is definable, then* $\dim(S) = \langle(d,0)\rangle$ *where* dim *is the classical dimension of definable subsets of K^m (as defined in 3.1 and 3.2).*

(iii) *If $f : S \to T$ is a definable map, where $S \subset (K^{alg})^m \times \Gamma^n$, $T \subset (K^{alg})^{m'} \times \Gamma^{n'}$ are definable, then*

$$\dim(f(S)) \leq \max_{k \geq 0}(\dim(S) + (-k,k)).$$

(iv) *If $f : S \to T$ is a definable bijection, then* $\dim(S) = \dim(T)$.

(v) $\dim(S \times T) = \dim(S) + \dim(T)$.

Proof. The last point to prove is (5), but one easily checks that the product of two cells is a cell (and the dimension of the resulting cell is the sum of the dimensions, were the dimension is taken as in 4.2.4). Hence (5) follows from 4.2.14 (5). □

Remark 4.2.19. If $X \subset (K^{alg})^m \times \Gamma^n$ is definable, for each integer i,

$$X(i) := \{x \in (K^{alg})^m \mid X_x \text{ is a definable set of } \Gamma^n \text{ of dimension } i\}$$

is easily seen to be a definable set of $(K^{alg})^m$. Then the mixed dimension that we have build can also be characterized by:

$$\dim(X) = \max_{\substack{X_i \neq \emptyset \\ i \geq 0}} \langle(\dim(X(i)), i)\rangle.$$

This follows from what precedes. We would like to mention that, with the right hand side definition, and 4.2.15, we could have proved the above theorem without mixed cells.

4.2.4 ℕ-valued dimension theory

As a corollary of the previous theorem we obtain a ℕ-valued dimension theory for definable sets of $(K^{alg})^m \times \Gamma^n$ which satisfies all expected properties.

Definition 4.2.20. Let $S \subset (K^{alg})^m \times \Gamma^n$ be a definable subset. We define

$$\dim_{\mathbb{N}} := \max_{(d_1,d_2) \in \dim(X)} d_1 + d_2.$$

Graphically, one can easily compute the dimension of some $\dim(X)$:

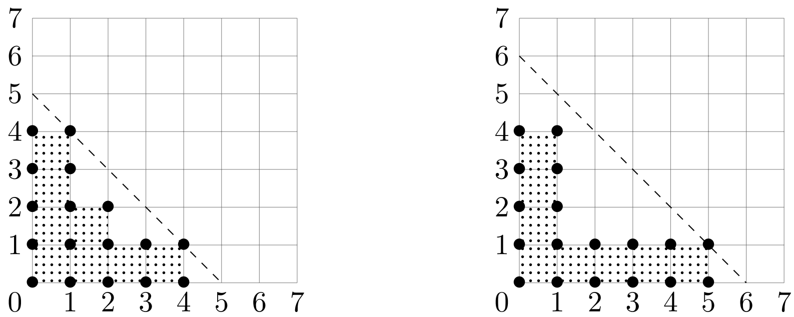


Figure 4.7: We consider the smallest integer d such that the dashed line $x+y=d$ is above the lower set $D=\dim(X)$. On the left for D_2 we obtain $d=5$, and on the right, for D_3 we obtain $d=6$.

Property (iii) below proceeds from the fact that if $(d_1,d_2)\in\mathbb{N}^2$ and $i\in\mathbb{N}$, $(d_1-i)+(d_2+i)\leq d_1+d_2$.

Theorem 4.2.21. *1. If $S\subset K^m$ is definable,* $\dim_{\mathbb{N}}(S)$ *is the* classical *dimension of S (as defined in sections 3.1 and 3.2).*

2. If $S\subset\Gamma^n$ is definable, $\dim_{\mathbb{N}}(S)$ *is the* classical *dimension of S (as defined in section 4.1.4).*

3. If $f:S\to T$ is a definable map where S (resp. T) is a definable set of $(K^{alg})^m\times\Gamma^n$ (resp. $K^{m'}\times\Gamma^{n'}$), then

$$\dim_{\mathbb{N}}(f(S))\leq\dim_{\mathbb{N}}(S).$$

4. If $f:S\to T$ is a definable bijection, then $\dim_{\mathbb{N}}(S)=\dim_{\mathbb{N}}(T)$.

5. $\dim_{\mathbb{N}}(S\times T)=\dim_{\mathbb{N}}(S)+\dim_{\mathbb{N}}(T)$.

4.3 Connection with other works

4.3.1 Tropicalization of maps between affinoid spaces

As we have explained in the introduction, this work started as an attempt to generalize the result [Duc12a, th 3.2]. This is achieved with

Theorem 4.3.1. *Let k be a non-Archimedean field. Let X be a compact k-analytic space, and let $f_1,\dots,f_n$ be n analytic functions defined on X. This defines a function*

$$\begin{array}{cccc} |f|: & X & \to & (\mathbb{R}_+)^n \\ & x & \mapsto & (|f_1(x)|,\dots,|f_n(x)|) \end{array}$$

Then $|f|(X)\cap(\mathbb{R}_+^*)^n$ is a Γ-rational polyhedral set of $(\mathbb{R}_+^*)^n$ of dimension $\leq\dim(X)$, where $\Gamma=\sqrt{|k^*|}$.

Proof. It is enough to prove the result when X is affinoid, so we will assume that X is an affinoid space.

First let K be an extension of k such that K is algebraically closed and complete, and $|K|=\mathbb{R}_+$. The ground field extension functor leads to the commutative diagram:

$$\begin{array}{ccc} X_K & & \\ \downarrow{\scriptstyle\pi} & \searrow^{|f_K|} & \\ X & \xrightarrow{|f|} & (\mathbb{R}_+)^n \end{array}$$

where π is surjective. Hence, $|f|(X) = |f_K|(X_K)$.

Secondly let us show that (under these assumptions on K), $|f|(X) = |f|(X(K))$. So let $(x_1, \ldots, x_n) \in |f|(X)$. We consider $U = \{x \in X \mid |f(x)| = (x_1, \ldots, x_n)\}$. Up to a permutation of the $f_i's$, we can assume that for some integer $0 \leq m \leq n$, $f_1(x) = \ldots = f_m(x) = 0$ and $f_{m+1}(x) \neq 0, \ldots, f_n(x) \neq 0$. Now, we denote by Z the Zariski closed subset of X defined by the vanishing locus of $f_1, \ldots, f_m$. Then, U is an affinoid domain of Z, which is non-empty by assumption. Moreover, since $|K| = \mathbb{R}_+$, this is a strict affinoid domain, so it contains a rigid point, z say. Hence $|f|(z) = (x_1, \ldots, x_n)$, and this shows that $|f|(X) = |f|(X(K))$.

Eventually we are reduced to show that if X is a k-affinoid space, $k \to K$ is a complete extension where $|K| = \mathbb{R}_+$, and K is algebraically closed, then $|f|(X(K)) \cap (\mathbb{R}_+^*)^n$ is a Γ-polyhedral set of $(\mathbb{R}_+^*)^n$ of dimension less or equal than $\dim(X)$.

That $|f|(X(K))$ is a definable set is now just an application of the analytic elimination theorem [Lip93, 3.8.2] (see 0.5.7). Indeed, by definition of an affinoid space, X can be identified with the Zariski closed subset of a closed polydisc, that is to say, we may find an integer N, and $g_1, \ldots, g_m \in k\langle T_1, \ldots, T_N\rangle$ such that $X(K)$ is identified with the Zariski closed subset $\{z = (z_1, \ldots, z_N) \in (K^\circ)^N \mid g_i(z) = 0,\ i = 1 \ldots m\}$. Moreover, the functions $f_1, \ldots, f_n$ of X arise as functions of $k\langle T_1, \ldots, T_N\rangle$, so that we will see the $f_i's$ as elements of $k\langle T_1, \ldots, T_N\rangle$. In addition, the f_i's induce a map $f : X \to \mathbb{A}_k^{n,an}$. We can easily reduce to the case $f : X \to \mathbb{B}^n$. Now,

$$f(X(K)) =$$
$$\{(x_1, \ldots, x_n) \in (K^\circ)^n \mid \exists (z_1, \ldots, z_N) \in (K^\circ)^N \text{ such that } g_i(z) = 0,\ i = 1 \ldots m, f_j(z) = x_j,\ j = 1 \ldots n\}$$

and this is a subanalytic set of $(K^\circ)^n$ defined over k, of dimension $\leq d$ according to lemma 3.1.27. According to theorem 4.2.21, $\mathrm{Trop}(f(X(K))$ is then a definable set of $(\mathbb{R}_+)^n$ of dimension $\leq d$, hence $\mathrm{Trop}(f(X(K)) \cap (\mathbb{R}_+^*)^n$ is also a definable set of $(\mathbb{R}_+^*)^n$ of dimension $\leq d$.

Finally, $\mathrm{Trop}(f(X(K)) \cap (\mathbb{R}_+^*)^n$ is a closed subset of $(\mathbb{R}_+^*)^n$. Indeed X is compact, $\mathrm{Trop} \circ f$ is continuous and remind that

$$\mathrm{Trop}(f(X(K)) = \mathrm{Trop}(f(X)).$$

Hence using the next (basic) lemma (after passing to subsets of $\mathbb{R}^n$ with a logarithm map), we conclude that $\mathrm{Trop}(f(X(K)) \cap (\mathbb{R}_+^*)^n$ is a polyhedral set of $(\mathbb{R}_+^*)^n$. ☐

Lemma 4.3.2. *Let us consider a nonempty set*

$$P = \bigcap_{j=1}^{m} \{(\gamma_i) \in \mathbb{R}^n \mid \lambda_j + \sum_{i=1}^{n} a_i^j \gamma_i \bowtie_j 0\}$$

where $\lambda_j \in \mathbb{R}$, $a_i^j \in \mathbb{Z}$ *and* $\bowtie_j \in \{<, \leq\}$. *Then*

$$\overline{P} = \bigcap_{j=1}^{m} \{(\gamma_i) \in \mathbb{R}^n \mid \lambda_j + \sum_{i=1}^{n} a_i^j \gamma_i \leq 0\}.$$

Note that if P is empty, it might happen that the set obtained replacing the $<$ by some $\leq$ is nonempty.

Proof. Let us denote by

$$Q = \bigcap_{j=1}^{m} \{(\gamma_i) \in \mathbb{R}^n \mid \lambda_j + \sum_{i=1}^{n} a_i^j \gamma_i \leq 0\}.$$

In one hand it is clear that $\overline{P} \subset Q$.

A simple calculation shows that if $p \in P$ and $q \in Q$ and $u \in]0,1]$, then $up+(1-u)q \in P$. Let then $q \in Q$ and let us pick $p \in P$. When $u \to 0$ we obtain that $q \in \overline{P}$, which shows that $Q \subset \overline{P}$. □

However, it has to be noticed that the situation studied by Ducros where the functions $f_i's$ are invertible has some specific properties. Namely, he proves [Duc12a, th 3.3], using the theory of reduction of germs developed by Temkin [Tem00, Tem04] that the germs of a the tropicalization map stabilize:

Theorem 4.3.3. *[Duc12a, th 3.3] Let X be a k-analytic space, $f = (f_i) : X \to \mathbb{G}_m^n$ an analytic map, and $|f| : X \to \mathbb{R}_+^n$ the induced tropicalization map. If $x \in X$, $\xi = |f|(x)$, there exists a neighbourhood U of x such that for all V neighbourhood of x such that $V \subset U$, the germs $(|f|(U), \xi)$ and $(|f|(V), \xi)$ are equal.*

If x is a rigid point, then the germ in question is a singleton (because the functions f_i are invertible, hence $|f_i|$ is locally constant around a rigid point). If we allow the functions f_i to vanish, the situation is different, and the germs of $|f|$ do not stabilize anymore:

Example 4.3.4. Let $f : \mathbb{B}_k^2 \to \mathbb{A}_k^{2,an}$ defined by $f(x,y) = (x, xy)$. Then, if $0 < r \leq 1$, $|f|(\mathbb{B}_r^2) = \{(u,v) \in \mathbb{R}_+^2 \mid u \leq r, v \leq ur\}$, and one checks that the germs $\big(|f|(\mathbb{B}_r^2), (0,0)\big)$ do not stabilize when r tends to 0.

4.3.2 Tropicalization of algebraic varieties

We would like to mention how this work relates to other results.
We have the following avatar of Gubler's result:

Proposition 4.3.5. *Let k be a non-Archimedean complete field. Let $X \subset \mathbb{B}_k^n$ be an irreducible embedded affinoid space of dimension d. Then Trop$(X \cap \mathbb{G}_{m,K}^n)$ is a polyhedral set of pure dimension d.*

Gubler's result implies that $\text{Trop}(X \cap \mathbb{G}_{m,K}^n)$ is a locally finite union of polytopes of dimension d. But according to theorem 4.3.1, this is in fact a finite union.

The mixed cell decomposition theorem has also the following consequence:

Proposition 4.3.6. *Let K be an algebraically closed non-Archimedean field, B an algebraic variety over K and X a closed subscheme of $B \times_K \mathbb{G}_{m,K}^n$. For each $b \in B(K)$, let $X_b \subset \mathbb{G}_{m,K}^n$ by the fibre of b. Then the family Trop$(X_b)_{b \in B(K)}$ takes only a finite number of topological types.*

Appendix A

Complex semi-algebraic subsets are not stable under projection

Definition A.0.7. A subset $V \subseteq \mathbb{R}^n$ is called a **real semialgebraic** subset if V is a finite boolean combination[1] of subsets of the form $\{x \in \mathbb{R}^n \mid f(x) > 0\}$ where $f \in \mathbb{R}[x_1, \dots, x_n]$.

We refer to [BCR98] and [vdD98] for general facts about semialgebraic sets.

Definition A.0.8. A subset $V \subseteq \mathbb{C}^n$ is called a **complex semi-algebraic** subset if V is a finite boolean combination of subsets of the form $\{z \in \mathbb{C}^n \mid |f(z)| < |g(z)|\}$ where $f, g \in \mathbb{C}[z_1, \dots, z_n]$.

In these definitions, we could also allow subsets defined with $\leq$ and $=$, since they can be obtained from $>$ and the boolean operators $^c, \cap$.

We can naturally identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$: if $z_1, \dots, z_n$ are the n complex coordinates of $\mathbb{C}^n$ and if we set $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$, then this identification is given by

$$\begin{array}{cccc} \varphi : & \mathbb{C}^n & \to & \mathbb{R}^{2n} \\ & (z_1, \dots, z_n) & \mapsto & (x_1, \dots, x_n, y_1, \dots, y_n) \end{array}$$

If $x, x', y, y' \in \mathbb{R}$, the following equivalence holds

$$|x + iy| < |x' + iy'| \Leftrightarrow x^2 + y^2 < x'^2 + y'^2.$$

It easily implies:

Lemma A.0.9. *Let V be complex semialgebraic subset of $\mathbb{C}^n$. Then, identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ through φ, V is a real semi-algebraic subset of $\mathbb{R}^{2n}$.*

Lemma A.0.11 and A.0.12 will clarify to what extent the converse is false.

It is easily seen that a complex semi-algebraic subset can be written as a finite union of sets of the form :

$$\left(\bigcap_{i=1}^{n} \{z \in \mathbb{C}^n \mid |f_i(z)| < |g_i(z)|\} \right) \bigcap \left(\bigcap_{j=1}^{m} \{z \in \mathbb{C}^n \mid |F_j(z)| = |G_j(z)|\} \right)$$

with $f_i, g_i, F_j, G_j \in \mathbb{C}[z_1, \dots, z_n]$, and m (resp. n) being possibly zero, which would mean that there would be only $<$ (resp. $=$) . We'll call such an intersection a *basic complex semi-algebraic subset* (the equalities here are assumed to be strict).

1. By finite boolean combination, we mean using finitely many times the symbols $\cap$, $\cup$ and ^{c}.

Remark A.0.10. In the definition of real semialgebraic subset of $\mathbb{R}^n$, if we replace inequalities $f > 0$ by $|g| > |h|$ (with g, h real polynomials) we get the same definition.
Indeed in one hand $|g| > |h|$ is equivalent to $g^2 - h^2 > 0$. One the other hand, $f > 0$ is equivalent to $|f + 1| > |f - 1|$.

Lemma A.0.11. *Let V be a real semialgebraic subset of $\mathbb{C}^n$ (we mean by that a real semialgebraic subset of $\mathbb{R}^{2n}$). Then there exist $m \in \mathbb{N}$, W a complex semialgebraic subset of $\mathbb{C}^{2n}$ such that $V = \psi(W)$ where ψ is a complex polynomial map $\psi : \mathbb{C}^{2n} \to \mathbb{C}^n$.*

Proof. The key point is to remark that inside $\mathbb{C}$,

$$\mathbb{R} = \{z \in \mathbb{C} \mid |z - i| = |z + i|\}. \tag{A.1}$$

We then write the k-th complex coordinate of $\mathbb{C}^n$

$$z_k = x_k + iy_k.$$

Let us suppose that $f(x_1, \dots, x_n, y_1, \dots, y_n)$) and $g(x_1, \dots, x_n, y_1, \dots, y_n)$ are two polynomials (in 2n real variables) such that

$$V = \{(x_1 + iy_1, \dots x_n + iy_n) \in \mathbb{C}^n \mid |f(x_1, \dots, x_n, y_1, \dots, y_n)| < |g(x_1, \dots, y_n)|\}.$$

We now consider $\mathbb{C}^{2n}$ with the complex coordinates $X_1, \dots, X_n, Y_1, \dots, Y_n$. Then we define

$$W_1 := \{(X_1, \dots, X_n, Y_1, \dots, Y_n) \in \mathbb{C}^{2n} \mid |X_k - i| = |X_k + i| \text{ and } |Y_k - i| = |Y_k + i| \text{ , } k = 1 \dots n\}$$

Then by the previous remark (A.1), W_1 is the set of points of $\mathbb{C}^{2n}$ with real coordinates.
Let us set

$$W_2 := \{(X_1, \dots, X_n, Y_1, \dots, Y_n) \in \mathbb{C}^{2n} \mid |f(X_1, \dots, Y_n)| < |g(X_1, \dots, Y_n)|\},$$
$$W := W_1 \cap W_2.$$

Eventually, let

$$\begin{array}{cccc} \psi : & \mathbb{C}^{2n} & \to & \mathbb{C}^n \\ & (X_1, \dots, X_n, Y_1, \dots, Y_n) & \mapsto & (X_1 + iY_1, \dots, X_n + iY_n) \end{array}$$

Then $\psi(W) = V$. Using remark A.0.10 and making use of finite boolean combination we can perform the proof. □

Lemma A.0.12. *Let $\mathcal{H} = \{x + iy \in \mathbb{C} \mid y^2 - (\cos(1)x)^2 = 1\}$. Then $\mathcal{H}$ is not a complex semi-algebraic subset of $\mathbb{C}$.*

Proof. Let us assume that $\mathcal{H}$ is a complex semialgebraic subset of $\mathbb{C}$, and let us decompose it in

$$\mathcal{H} = \bigcup_{k=1 \dots N} V_k$$

where the V_k's are basic complex semialgebraic subsets. One of the V_k's, let us call it V, must contain infinitely many points of $\mathcal{H}$. Let us write it

$$V = \left(\bigcap_{i=1}^{n} \{z \in \mathbb{C} \mid |F_i(z)| < |G_i(z)|\} \right) \bigcap \left(\bigcap_{j=1}^{m} \{z \in \mathbb{C} \mid |f_j(z)| = |g_j(z)|\} \right).$$

Clearly, V can not be open in $\mathbb{C}$, since it is contained in $\mathcal{H}$ whose interior is empty, so in the expression of V there must be an equality $|f(z)| = |g(z)|$ (i.e. $m \geq 1$), which defines a strict real Zariski-closed subset of $\mathbb{R}^2 \simeq \mathbb{C}$. If we call

$$W = \{z \in \mathbb{C} \mid |f(z)| = |g(z)|\},$$

by assumption, W is a proper real Zariski-closed subset of $\mathbb{R}^2 \simeq \mathbb{C}$ which contains infinitely many points of $\mathcal{H}$. Now, $\mathcal{H}$ is an irreducible real Zariski closed subset of $\mathbb{R}^2$ of dimension 1, so $\mathcal{H} \subseteq W$.

Since $f, g \neq 0$ (otherwise W would be finite), there is an asymptotic Taylor series:

$$\frac{f(z)}{g(z)} \underset{|z|\to\infty}{\sim} \beta z^k$$

with $\beta \in \mathbb{C}^*$ and $k \in \mathbb{Z}$. But since $\mathcal{H}$ is unbounded and for $z \in \mathcal{H}$, $|f(z)| = |g(z)|$, it follows that $k = 0$ and $|\beta| = 1$. Moreover, since $W \neq \mathbb{C}$, $|\frac{f}{g}|$ is not constant. Hence, dividing f by β, we obtain an asymptotic Taylor series of the form

$$\frac{f(z)}{g(z)} \underset{|z|\to\infty}{=} 1 + \frac{\alpha}{z^p} + o\left(\frac{1}{|z^p|}\right)$$

with $\alpha \in \mathbb{C}^*$ and $p \in \mathbb{N}^*$.

By assumption, for all $z \in \mathcal{H}$ where g does not vanish, $|\frac{f(z)}{g(z)}|^2 = 1$. When passing to the absolute value, the asymptotic series gives us for $z \in \mathcal{H}$,

$$1 + 2\Re(\alpha z^{-p}) + \circ(|z|^{-p}) = 1,$$

i.e.

$$\Re(\alpha z^{-p}) = \circ(|z|^{-p}).$$

Now, $\mathcal{H}$ has two asymptotic branches $\mathcal{D}_1 = \{y = -\cos(1)x\}$, and $\mathcal{D}_2 = \{y = \cos(1)x\}$. When $|z| \to \infty$, on $\mathcal{D}_1$, $z \sim e^{-i}|z|$, and on $\mathcal{D}_2$, $z \sim e^{i}|z|$. On these two branches, the Taylor series we have found implies that:

$$\Re(\alpha(e^{-i}|z|)^{-p}) = |z|^{-p}\Re(\alpha e^{ip}) = \circ(|z|^{-p}) \text{ on } \mathcal{D}_1$$
$$\Re(\alpha(e^{i}|z|)^{-p}) = |z|^{-p}\Re(\alpha e^{-ip}) = \circ(|z|^{-p}) \text{ on } \mathcal{D}_2.$$

This implies that $\Re(\alpha e^{ip}) = \Re(\alpha e^{-ip}) = 0$ which is impossible since π is irrational. This contradicts the fact that $\mathcal{H}$ is a complex semialgebraic subset. $\square$

Corollary A.0.13. *Complex semialgebraic subsets are not stable under projection.*

Proof. Indeed, otherwise, according to lemma A.0.11, the real semialgebraic subsets of $\mathbb{C}^n$ would be complex semialgebraic, but the hyperbola $\mathcal{H}$ gives a counter-example. $\square$

List of Figures

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Glossary of Notations

D	division operator	10
$\mathcal{A}\langle\langle D\rangle\rangle$	overcongergent D-functions	10
$k\langle\langle X_1,\dots,X_n\rangle\rangle$	overconvergent functions on the disc	10
D_0	division operator	11
D_1	division operator	11
Rig_k	Category of rigid k-spaces	21
$\mathcal{O}_X(U)$	algebra of analytic functions on the affinoid domain U	22
$\mathcal{A}$	k-affinoid algebra	22
$\mathcal{A}\{r^{-1}X\}$		22
$\mathrm{Sp}(\mathcal{A})$	affinoid space (as a rigid k-space)	22
$k\langle X_1,\dots,X_n\rangle$	k-algebra of convergent power series	22
$k\{X_1,\dots,X_n\}$	k-algebra of convergent power series (Berkovich notation)	22
$\mathbb{B}$, $\mathbb{B}_k$	closed unit dic	23
$\mathbb{B}^n$, $\mathbb{B}^n_k$	closed polydisc	23
$\mathcal{H}(x)$	completed residue field	24
$\mathcal{M}(\mathcal{A})$	Spectrum of a Banach algebra	23
$\mathcal{X}^{\mathrm{an}}$	analytification of a k-scheme of finite type	25
$\mathrm{Int}(Y)$	interior of Y	26
$\mathrm{Int}(Y/X)$	relative interior	26
$\partial(Y/X)$	boundary of a morphism	26
$S_{m,n}$, $S_{m,n}(E,K)$		38
D-function	algebra of overconvergent D-functions	39
$\mathcal{L}^D_{an}$-formula	formula in the analytic language $\mathcal{L}^D_{an}$	40
π	uniformizer of k when it is a DVR	44
$H^q_c(X,F)$	compactly supported cohomology group in k-analytic spaces	47
$S(X)$	catégorie des faisceaux étales sur un espace k-analytique X	47
$(Y,T) \overset{\varphi}{\dashrightarrow} (X,S)$	constructible datum	54
$\mathcal{A}\langle\langle D\rangle\rangle$	overconvergent D-functions, in the framework of Berkovich spaces	78

Index of notations