ADIC SPACES

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Contents

1. Affinoid rings and spaces

We fix k a non Archimedean field, that is to say, k is a topological field, whose topology can be defined by a rank 1 valuation, and R will be a topological ring. In fact in the context of perfectoid spaces and rigid geometry, the topological ring R will always be a topological k-algebra, i.e. the topology of R will be compatible with the topology of k. Actually, most of the time, R will even be a normed k-algebra. Recall that if R is a topological ring, we say that an ideal I defines the topology of R, if $\{I^n\}_{n\in\mathbb{N}}$ is a basis of neighborhood of 0, and in that case R is called an adic ring.

Definition 1.1 ([Hub93]). Let R be a topological ring.

(1) The ring R is called f-adic if there exists R_0 an open subring such that the topology of R_0 is defined by a finitely generated ideal I of R_0.
(2) R is called a Tate ring if it is a f-adic ring and if there exists an invertible element which is topologically nilpotent.

Definition 1.2. A subset $M \subset R$ is said to be bounded if for all neighborhood of 0, U, there exists V a neighborhood of 0 such that $M.V \subseteq U$. An element $a \in R$ is said to be power-bounded if $\{a^n \mid n \in \mathbb{N}\}$ is bounded. We denote by R° the set of power-bounded elements.

Remark 1.3. If R is a normed k-algebra, one checks that $a \in R$ is power-bounded if and only if $\{\|a^n\| \mid n \in \mathbb{N}\}$ is bounded.

Definition 1.4. (1) A valuation on R is a map $v : R \to \Gamma_0 = \Gamma \cup \{0\}$ where Γ is a totally ordered commutative abelian group (noted multiplicatively), such that

$$\begin{aligned} v(ab) &= v(a)v(b) \\ v(a+b) &\leq \max(v(a), v(b)) \\ v(0) &= 0 \text{ and } v(1) = 1. \end{aligned}$$

The order on Γ_0 is defined by the order of Γ and the fact that 0 is a minimum element. Moreover, we set $0 \cdot \gamma = 0$ for all $\gamma \in \Gamma$. To simplify notations, we will often denote valuations by $|\cdot|$. We will assume most of the time that the subgroup of Γ generated by $v(R) \setminus \{0\}$ is Γ.

(2) A valuation $|\cdot| : R \to \Gamma_0$ is said to be a continuous valuation if for all $\gamma \in \Gamma$, there exists U a neighborhood of 0 such that $|U| \subseteq [0, \gamma[= \{\alpha \in \Gamma_0 \mid \alpha < \gamma\}$.

(3) The subgroup of Γ generated by $|R| \setminus \{0\}$ is called the value group of $|\cdot|$ and is denoted by $\Gamma_{|\cdot|}$.

(4) Two valuations v and w on R are called equivalent if there exists an isomorphism of totally ordered groups, $\alpha : \Gamma_v \simeq \Gamma_w$ such that for all $a \in R$, $w(a) = \alpha(v(a))$.

If $|\cdot|$ is a valuation, $\text{supp}(|\cdot|) = \{a \in R \mid |a| = 0\}$ is a prime ideal of R and $|\cdot|$ induces a valuation on the fraction field $K = Frac\big(R/\text{supp}(|\cdot|)\big)$. One can check that two valuations v and w are equivalent if and only if $\text{supp}(v) = \text{supp}(w)$ and the valuation rings they define on K are the same. This is also equivalent to say that for all $a, b \in R$, $v(a) \leq v(b)$ if and only if $w(a) \leq w(b)$.

Definition 1.5. An affinoid ring is given by a pair (R, R^+) where R is a f-adic ring, and $R^+ \subseteq R^\circ$ is an open and integrally closed subring of R. A morphism f of affinoid rings between (R, R^+) and (S, S^+) is a continuous morphism $f : R \to S$ such that $f(R^+) \subseteq S^+$.

In [Hub93] affinoid rings are denoted by $(A^\triangleright, A^+)$ and subrings of $A^\triangleright$ having the same property as A^+ are called rings of integral elements.

Remark 1.6. The definitions of [Hub96] ,[Sch12],[Fon13] might seem different but actually agree. Indeed let R be a topological k-algebra, the following propositions are equivalent :

(1) R is f-adic
(2) R is a Tate ring
(3) There exists a subring R_0 such that aR_0, $a \in k^\times$ forms a basis of open neighborhoods of 0.

Remark that (3) is the definition of a Tate k-algebra in [Sch12, 2.6]. If R is in fact a normed k-algebra, R is automatically a Tate k-algebra.

Definition 1.7. Let (R, R^+) be an affinoid ring. One defines

$$\begin{aligned} X &= \text{Spa}(R, R^+) \\ &:= \{\text{continuous valuations } |\cdot| : R \to \Gamma_0 \text{ such that } |R^+| \leq 1\} / \simeq \end{aligned}$$

where $\simeq$ is the equivalence relation of Definition 1.4 (4). We equip X with the topology generated by the open subsets $\{|\cdot| \in X \mid |a| \leq |b| \neq 0\}$ where $a, b \in R$.

Notation 1. If $x \in \mathrm{Spa}(R, R^+)$, then x is a valuation $x : R \to \Gamma_0$. For $f \in R$, we will set

$$|f(x)| := x(f).$$

Definition 1.8 ([Hoc69])**.** A topological space X is spectral if X is quasi-compact, has a basis of topology made by quasi-compact open which is stable under finite intersection, and such that every irreducible closed subset has a unique generic point.

In [Hoc69] it is proved that a topological space X is spectral if and only if it is homeomorphic to $\mathrm{Spec}(B)$ for some ring B.

Proposition 1.9. *Let (R, R^+) be an affinoid ring. Then $X := Spa(R, R^+)$ is a spectral space. Let us define a rational subset of X as*

$$U(\frac{f_1, \dots, f_n}{g}) = \{x \in X \bigm| |f_i(x)| \leq |g(x)| \neq 0 \ i = 1 \dots n\}$$

where the f_i's define an open ideal of R. Then the rational subsets form a basis of neighborhood of X which is stable under finite intersection.

Spa then defines a functor from the category of affinoid ring to the category of topological spaces. When R is a Tate ring, R is the only open ideal of R, so in that case, saying that the f_i's generate an open ideal is equivalent to saying that the f_i's generate R, and in that case

$$\{x \in X \bigm| |f_i(x)| \leq |g(x)| \neq 0 \ i = 1 \dots n\} = \{x \in X \bigm| |f_i(x)| \leq |g(x)| \ i = 1 \dots n\}.$$

Example 1.10.

(1) If $\mathcal{A}$ is an affinoid k-algebra (in the sense of rigid geometry [BGR84]) i.e. a quotient of the Tate algebra $k\langle T_1, \dots, T_n\rangle$, then $(\mathcal{A}, \mathcal{A}^\circ)$ is an affinoid ring[1] in the above sense. The following result holds:

Theorem 1.11 ([Hub93, Corollary 4.4])**.** *Let $\mathcal{A}$ be an affinoid k-algebra. Then the topos associated to $X = Spa(\mathcal{A}, \mathcal{A}^\circ)$ is equivalent to the topos of the rigid space $Max(\mathcal{A})$.*

(2) If R is a Noetherian ring, I is an ideal of R such that R is complete for the topology defined by I, then (R, R) is an affinoid ring.

(3) Let $R = k\langle T^{1/p^\infty}\rangle$ and $R^+ = k^\circ\langle T^{1/p^\infty}\rangle$. Then (R, R^+) is an affinoid ring. If $x = (x_i)_{i \geq 1}$ is a sequence of points of k^{alg} such that $x_{i+1}^p = x_i$ for all $i \geq 1$, and $x_1 \in (k^{\mathrm{alg}})^\circ$, then we can define a morphism of k-algebra $x : k\langle T^{1/p^\infty}\rangle \to k^{\mathrm{alg}}$ defined by $T^{1/p^n} \mapsto x_n$, and

$$\begin{array}{rccc} |\cdot|_x : & k\langle T^{1/p^\infty}\rangle & \to & \mathbb{R}_+ \\ & f & \mapsto & |f(x)| \end{array}$$

is a point of $\mathrm{Spa}(k\langle T^{1/p^\infty}\rangle, k^\circ\langle T^{1/p^\infty}\rangle)$.

2. (Pre)-Sheaves

Let $X = \mathrm{Spa}(R, R^+)$ and $U = U(\frac{f_1, \dots, f_n}{g})$ be a rational subset. We consider B the integral closure of $R^+[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ in $R[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$. Then, $(R[\frac{f_1}{g}, \dots, \frac{f_n}{g}], B)$ is

[1] One must check that $\mathcal{A}^\circ$ is integrally closed in $\mathcal{A}$.

an affinoid ring. We then take its completion $(R\langle \frac{f_1}{g},\dots,\frac{f_n}{g}\rangle, \hat{B})$.[2] By functoriality, one can define a morphism

$$\psi : \mathrm{Spa}(R\langle \frac{f_1}{g},\dots,\frac{f_n}{g}\rangle, \hat{B}) \to \mathrm{Spa}(R, R^+)$$

It fulfills the following universal property: for all complete affinoid ring (S, S^+) and $\varphi : (R, R^+) \to (S, S^+)$ a morphism of affinoid ring such that $\mathrm{Im}\ (\mathrm{Spa}(\varphi)) \subseteq U$ where $\mathrm{Spa}(\varphi) : \mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$, then φ factorizes uniquely through $(R\langle \frac{f_1}{g},\dots,\frac{f_n}{g}\rangle, \hat{B})$.

From this it follows that $R\langle \frac{f_1}{g},\dots,\frac{f_n}{g}\rangle$ and $\hat{B}$ depend only on U. We then set $\mathcal{O}_X(U) = R\langle \frac{f_1}{g},\dots,\frac{f_n}{g}\rangle$ and $\mathcal{O}_X^+(U) = \hat{B}$. In this way, one checks that $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are presheaves on the rational subsets of X. Now, if $W \subset X$ is an open subset, we set

$$\mathcal{O}_X(W) = \varprojlim_{U \subseteq W} \mathcal{O}_X(U)$$

where the limit is taken over the rational subsets $U \subseteq W$, and likewise

$$\mathcal{O}_X^+(W) = \varprojlim_{U \subseteq W} \mathcal{O}_X^+(U).$$

Remark that $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are presheaves of complete topological rings. One checks that for all $x \in X$, and $x \in U$ a rational subset, $x : R \to \Gamma$ can be extended to $x : \mathcal{O}_X(U) \to \Gamma$. So x can be extended to $\mathcal{O}_{X,x}$. It follows that $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $M_x = \{f \in \mathcal{O}_{X,x} \mid |f(x)| = 0\}$. We set $k(x) = \mathcal{O}_{X,x}/M_x$. So $k(x)$ is naturally equipped with a valuation: $f \to |f(x)|$ and we set $k^+(x)$ its valuation ring. For an open subset U:

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid |f(x)| \leq 1\ \forall x \in U\}.$$

In general, $\mathcal{O}_X$ is not a sheaf. The previous remark however shows that if it was a sheaf, then $\mathcal{O}_X^+$ would also be a sheaf. However

Definition 2.1. A topological ring is strongly Noetherian if for all $n \in \mathbb{N}$, $R\langle T_1, \dots, T_n\rangle$ is Noetherian [3].

Theorem 2.2 ([Hub94, Theorem 2.2])**.** *Let R be strongly Noetherian, and let $X := Spa(R, R^+)$. Then $\mathcal{O}_X$ is a sheaf.*

One then defines (V) as the category of locally ringed spaces $(X, \mathcal{O}_X)$, such that the sheaf $\mathcal{O}_X$ is a sheaf of topological rings, and such that for all $x \in X$, there is given an equivalence class of valuation v_x of the stalk $\mathcal{O}_{X,x}$. The morphisms in (V) must be compatible with all these data.

Definition 2.3. Let (R, R^+) be an affinoid ring and $X = \mathrm{Spa}(R, R^+)$. If $\mathcal{O}_X$ is a sheaf, we'll say it is an affinoid adic space (seen as an object of (V)). An adic space is an object of (V) which is locally an affinoid adic space.

Adic spaces fulfill most of the expected properties, such as

[2]One has to convince himself that the completion of arbitrary topological rings exists, and that the completion $(\hat{R}, \hat{R^+})$ of an affinoid ring (R, R^+) is still an affinoid ring.

[3]I do not know an example of a Banach Noetherian ring R which is not strongly Noetherian, but I guess some people do.

Proposition 2.4. *If X and $Spa(R,R^+)$ are adic spaces, then*

$$\underset{adic\ \ spaces}{Hom}(X, Spa(R,R^+)) \simeq \underset{affinoid\ rings}{Hom}\big((\hat{R},R^+),(\mathcal{O}_X(X),\mathcal{O}_X^+(X))\big).$$

Theorem 2.5. *There is a functor*

$$r_k : \{rigid\ spaces\ over\ k\} \to \{adic\ spaces\ over\ Spa(k,k^\circ)\}$$

obtained by gluing the functor $Max(\mathcal{A}) \mapsto Spa(\mathcal{A},\mathcal{A}^\circ)$. It is fully faithful and induces an equivalence of category :

$$\{rigid\ spaces\ over\ k\ quasi\text{-}separated\} \simeq \{quasi\text{-}separated\ adic\ spaces\ locally\ of\ finite\ type\ over\ Spa(k,k^\circ)\}$$

where finite type will be defined in the next section.

3. The closed disc and the affine line

We want to explain the classification of points in the affine line, when k is algebraically closed, the so called points of type 1,2,3,4 and 5. So we will assume that $k = k^{alg}$. Sometimes we will write k^{alg} to really emphasize on it.

3.1. **The closed disc *vs.* the affine line.** We set $\mathbb{B} = \mathrm{Spa}\big(k\langle T\rangle, k^\circ\langle T\rangle\big)$ and call it the closed unit disc. More generally, for $r \in |k^*|$, let $\mathbb{B}_r = \mathrm{Spa}\big(k\langle r^{-1}T\rangle, k^\circ\langle r^{-1}T\rangle\big)$. When $r \leq r'$, $\mathbb{B}_r$ is an rational subset of $\mathbb{B}_{r'}$, so we can glue the $\mathbb{B}_r$'s to form $\mathbb{A}_k^{\mathrm{ad}}$, which is the union of the $\mathbb{B}_r$'s.

Lemma 3.1. *$\mathbb{B}$ is in natural bijection with the set of valuations $v : k[T] \to \Gamma_0$ such that $v(T) \leq 1$ and such that the convex subgroup generated by $v(k^*)$ is Γ itself.*

Proof. One can restrict a continuous valuation $v : k\langle T\rangle \to \Gamma_0$ to $k[T]$. Let us prove that this restriction $v : k[T] \to \Gamma_0$ satisfies the condition that that the convex subgroup generated by $v(k^*)$ is Γ itself. Let $\gamma \in \Gamma$ and let us prove that γ is in the convex subgroup generated by $v(k^*)$. Considering γ^{-1} if necessary, we can assume that $\gamma \leq 1$. So by the continuity assumption on $v \in \mathbb{B}$, there exists an open neighborhood U of $0 \in k\langle T\rangle$ such that $v(U) \subset [0,\gamma[$. So in particular, there exists $\lambda \in k^*$ such that $v(\lambda) \leq \gamma$. This implies that $\gamma \leq v(\lambda^{-1})$.

Injectivity. Let $v, v' \in \mathbb{B}$ and let us assume that their restriction to $k[T]$ coincide. Then v and v' coincide on $k[T]$. This follows for instance from Weierstrass Preparation Theorem: if $f \in k\langle T\rangle$, then f has a factorization $f = P.u$ where $P \in k[T]$ and $u \in 1 + k^{\circ\circ}\langle T\rangle$. Sometimes u is called a multiplicative unit because u is invertible and $u, u^{-1} \in k^\circ\langle T\rangle$. In particular $v(u) \leq 1$ and $v(u^{-1}) \leq 1$, so $v(u) = 1$. Hence v and v' are determined by their restriction to $k[T]$.

Surjectivity. Let $v : k[T] \to \Gamma_0$ be a valuation such that $v(T) \leq 1$ and the convex subgroup generated by $v(k^*)$ is Γ. Let $f \in k\langle T\rangle$, and let f_n be a sequence of $k[T]$ which converges to f. Then either $v(f_n)$ tends to 0 in Γ_0, either it is stationary. This relies on the fact that if $\lambda_n \to 0$ in k, then $v(\lambda_n) \to 0$ in Γ_0 (see Remark 3.2 (2) for a counter example in general). In any case $v(f_n)$ has a limit, it does not depend on the choice of the sequence f_n, and we set $v(f) := \lim_n v(f_n)$. One checks that this defines a valuation $\tilde{v}$ on $k\langle T\rangle$. Let us prove that it $\tilde{v}$ is continuous. Let $\gamma \in \Gamma$ and let us prove that $\tilde{v}^{-1}([0,\gamma[)$ is a neighborhood of 0. Since the convex subgroup generated by $v(k^*)$ is Γ, there exists some $\lambda \in k^*$ such that $v(\lambda) \leq \gamma$. Then $v(\lambda k^\circ\langle T\rangle) \subset [0,\gamma[$ and $\lambda k^\circ\langle T\rangle)$ is a neighborhood of 0. □

Remark 3.2. (1) The use of Weierstrass Preparation is not optimal. In particular, this could generalize to $(\mathbb{A}_k^n)^{\mathrm{ad}}$. Let us mention that according to [Hub93, Proposition 3.9], there is a homeomorphism $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle) \simeq \mathrm{Spa}(k[T], k^\circ[T])$.

(2) Let us consider the ordered group $\mathbb{R}_+^* \times \varepsilon^{\mathbb{Z}}$, with $\varepsilon < r$ for any $r \in \mathbb{R}_+$. Then the sequence $\frac{1}{n}$ does not tend to 0 in Γ. We can define a valuation, $v : P = \sum a_i T^i \in k[T] \mapsto \max |a_i| T^i$. This max is uniquely attained for the smallest index i such that $a_i \neq 0$. Then v is a valuation, which satisfies $v(T) \leq 1$, nut v is not continuous.

One last comment : let $k \to K$ be some extension of valued fields, and let us denote by $|\cdot|_K : K \to \Gamma_0$ the norm on K. Let us assume that the convex subgroup generated by $|k^*|_K$ is Γ. Any $x \in K$ induces a point of $\mathbb{A}_k^{1,\mathrm{ad}}$ defined by $P \in k[T] \mapsto |P(x)|_K$. And any point of $\mathbb{A}_k^{1,\mathrm{ad}}$ arises in this way. The above construction induces a continuous map $K \to \mathbb{A}_k^{1,\mathrm{ad}}$. This map is never injective, except when $K \subset k^{p^{-\infty}}$, the perfect closure of k. It might be surjective but K has to be big. In particular, when $K = \widehat{k^{\mathrm{alg}}}$ it is not surjective. And it it never bijective. Let us finally remark that the existence of non rigid points is very specific to the non-Archimedean situation. If k is any non-Archimedean field, one can extend, and in many ways, the norm of k to $k(T)$. This is not the case in the Archimedean setting : one can not extend the Archimedean norm $|\cdot|_\infty$ of $\mathbb{C}$ to $\mathbb{C}(T)$ because according to Gelfand-Mazur Theorem, the only $\mathbb{C}$-Banach field is $\mathbb{C}$.

3.2. **Points of type 1 to 5.** We assume that $k = k^{\mathrm{alg}}$. We want to describe 5 families of points which appear in $\mathbb{A}_k^{1,\mathrm{ad}}$. In the next subsection, we will show that these are the only ones.

3.2.1. *Type 1.* They correspond to maximal ideals of $k[T]$, and are in canonical bijection with the orbits of k^{alg} under the action of $\mathrm{Gal}(k^{sep}/k)$ according to the Nullstellensatz. If $x \in k^{\mathrm{alg}}$, the associated valuation is simply given by $P \in k[T] \mapsto |P(x)| \in \mathbb{R}_+$.

3.2.2. *Type 2 and 3.* We start with the main example, the so called Gauss point that we will denote by η.

$$\eta : \begin{array}{ccc} k[T] & \to & \mathbb{R}_+ \\ \sum_i a_i T^i & \mapsto & \max_i |a_i| \end{array}$$

Fact. η defines indeed a valuation.

Proof. The only thing to check is that for $P = \sum a_i T^i$ and $Q = \sum b_i T^i$, if $PQ = \sum c_i T^i$ then $\max(|a_i|).\max(|b_i|) = \max(|c_i|)$. Up to a normalization of the coefficients, we can assume that $\max(|a_i|) = \max(|b_i|) = 1$, which is equivalent to require that $P, Q \in k^\circ[T]$ and that their reduction in $\tilde{k}[T]$ is nonzero. But then PQ is also in $K^\circ[T]$ and its reduction is also nonzero, so $\eta(PQ) = 1$. □

Remind that the norm on k extends uniquely to k^{alg}.

Fact. For $P \in k[T]$, $\eta(P) = \max\limits_{z \in (k^{\mathrm{alg}})^\circ} |P(z)|$.

Proof. Up to a normalization, we can still assume that $\eta(P)=1$, i.e. that $P\in k^\circ[T]$ and that its reduction $\tilde{P}[T]$ is nonzero. Then for $z\in (k^{\mathrm{alg}})^\circ$, $|P(z)|\leq 1$. Finally, let us pick some $\tilde{\lambda}\in\tilde{k}$ such that $\tilde{P}(\lambda)\neq 0$, and and some $\lambda\in(k^{\mathrm{alg}})^\circ$ whose reduction is $\tilde{\lambda}$. Since $\widetilde{P(\lambda)}\neq 0$, it follows that $|P(\lambda)|=1$. □

Now, let us consider $r\in\mathbb{R}_+^*$, and define

$$\begin{array}{cccc}\eta_r: & k[T] & \to & \mathbb{R}_+\\ & \sum_i a_iT^i & \mapsto & \max_i |a_i|r^i\end{array}$$

This generalizes the above construction because $\eta=\eta_1$. For $c\in k^{\mathrm{alg}}$ let us set $B(c,r):=\{z\in k^{\mathrm{alg}}\;\big|\;|z-c|\leq r\}$ the closed ball in k^{alg} of center c and radius r.

Fact. For $r\in\mathbb{R}_+^*$, η_r defines a valuation. For $P\in k[T]$, $\eta_r(P)=\sup_{z\in B(0,r)}|P(z)|$.

Proof. If $r\in\sqrt{|k^*|}$ the above proof works, and we can write max instead of sup. In general, to show that η_r is a valuation, let $P=\sum a_iT^i, Q=\sum b_iT^i\in k[T]$ and let $PQ=\sum c_iT^i$. Then the ultrametric property shows that $\eta_r(PQ)\leq\eta_r(P)\eta_r(Q)$. On the other hand, an easy calculation shows that if i (resp. j) is the smallest index such that $|a_ir^i|=\eta_r(P)$ (resp. $|b_jr^j|=\eta_r(Q)$) then $|c_{i+j}|r^{i+j}=|a_i|r^i|b_j|r^j$ which proves that $\eta_r(PQ)\geq\eta_r(P)\eta_r(Q)$, so η_r is a valuation. To prove the characterization of η_r in terms of sup, one can remark that if $P\in k[T]$ then $s\in\mathbb{R}_+^*\mapsto\eta_s(P)$ is continuous, thus $\lim\limits_{s\to r,\ s<r}\eta_s(P)=\eta_r(P)$. If $r\notin\sqrt{|k|}$ then

$$B(0,r)=\cup_{s<r}B(0,s)=\bigcup_{s\in\sqrt{|k^\times|},\ s<r}B(0,s).$$

Beware that if $r\in\sqrt{|k^\times|}$, then $\cup_{s<r}B(0,s)\subsetneq B(0,r)$. Finally

$$\eta_r(P)=\lim_{\substack{s\in\sqrt{|k^\times|}\\ s<r}}\eta_s(P)=\lim_{\substack{s\in\sqrt{|k^\times|}\\ s<r}}\sup_{z\in B(0,s)}|P(z)|=\sup_{z\in B(0,r)}|P(z)|.$$

□

More generally, if $c\in k, r\in\mathbb{R}_+^*$, we set

$$\begin{array}{cccc}\eta_{c,r}: & k[T] & \to & \mathbb{R}_+\\ & \sum_i a_i(T-c)^i & \mapsto & \max_i |a_i|r^i\end{array}$$

Hence $\eta_{c,r}$ is obtained from η_r by a change of variables: $\eta_{c,r}\big(P(T)\big)=\eta_r\big(P(T+c)\big)$. It then follows from the above facts that

Fact. $\eta_{c,r}$ defines a valuation. For $P\in k[T]$, $\eta_{c,r}(P)=\sup_{z\in B(c,r)}|P(z)|$. For two choices $(c,r),(c',r')$ then $\eta_{c,r}=\eta_{c',r'}$ if and only if $B(c,r)=B(c'r')$ which is also equivalent to say that $r=r'$ and $|c-c'|\leq r$.

If $r\in\sqrt{|k^\times|}$, $\eta_{r,c}$ is called a type 2 point. If $r\notin\sqrt{|k^\times|}$, $\eta_{r,c}$ is called a type 3 point.

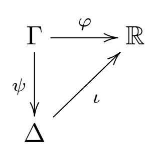
3.2.3. *Type 4.* A non-Archimedean extension $k \to K$ is said to be an immediate extension if the associated extensions of residue fields $\tilde{k} \to \tilde{K}$ and value group $|k^*| \to |K^*|$ are trivial. If $x \in K \setminus k$ we will denote by $v_x : P \in k[T] \mapsto |P(x)|_K$ the associated point of $\mathbb{A}_k^{1,\mathrm{ad}}$ and will say it is a type 4 point. For instance, $\widehat{\mathbb{C}((t))^{\mathrm{alg}}} \to \mathbb{C}((t^{\mathbb{Q}}))$ is an immediate extension. One can attach a type 4 point to the element

$$\sum_{n \in \mathbb{N}^*} t^{\left(\frac{-1}{n}\right)} \in \mathbb{C}((t^{\mathbb{Q}})) \setminus \widehat{\mathbb{C}((t))^{\mathrm{alg}}}.$$

If $x \in K \setminus k$ as above, let us consider $\mathcal{B}$ the set of closed ball in K with center in k which contain x. That is to say we consider balls $B = \{z \in K \mid |z - c| \leq r\}$ such that $c \in k$, $r \in \mathbb{R}_+^*$ and $x \in B$. Then $\mathcal{B}$ is a totally ordered set for inclusion because two balls of $\mathcal{B}$ with the same radius r have a common point (namely x), so are equal. It follows that $\mathcal{B} = \{B_i\}_{i \in I}$ where I is an interval of $\mathbb{R}_+^*$ of the form $\ldots, +\infty[$. It easy to see that I must be of the form $]a, +\infty[$ for some $0 < a$. If a was 0, by completeness of k we would conclude that $x \in k$. And the interval must be open around a: indeed let us assume that $I = [a, +\infty[$, and let $c \in k$ such that $|x - c| \leq a$. Then, a must be equal to $|x - c|$, otherwise $b := |x - c| < a$ and we should have $b \in I$. So, since $a = |x - c|$, up to a rescalling, we can assume that $a = 1$. Then since $|x - c| \leq 1$, since $\tilde{k} = \tilde{K}$ we can even find some $d \in k$ such that $|x - d| < 1$ which is a contradiction.

3.2.4. *Type 5.* Points of type 1 to 4 are rank 1 valuations. This means that their value group Γ is a totally ordered abelian group of rank 1. The rank d of a totally ordered abelian group Γ is the smallest integer d such that Γ can be embedded as a totally ordered subgroup of $(\mathbb{R}, +, <)^d$ equipped with the lexicographic order. This is equivalent to say that there is no sequence $G_0 \subsetneq G_1 \subsetneq G_2 \ldots \subsetneq G_{d+1}$ of convex subgroup of Γ.

The easy thing with rank 1 valuations, is that one does not have to define in advance the value group Γ. We were starting with some norms $|\cdot| : k \to \Gamma_0 \subset \mathbb{R}_+$, and we have extended the valuation staying in $\mathbb{R}_+$. Let us give a complicated reason for this: if $\Gamma \xrightarrow{\varphi} (\mathbb{R}, +)$ and $\Gamma \xrightarrow{\psi} \Delta$ are injections of totally ordered abelian groups, and if $\Gamma \neq \{0\}$ and Δ has rank 1, then there is a unique morphism of totally ordered abelian groups $\iota : \Delta \to \mathbb{R}$ making the following diagram commute



To define a rank 2 valuation, we first have to define a value group Γ of rank 2. Let us give a naive description of this Γ. It will contain $\mathbb{R}_+^*$. It will also contain some element $\gamma \in \Gamma \setminus \mathbb{R}_+^*$ which is smaller than 1 but infinitesimally closed to 1. One might think of γ as some 1_-. More precisely, $\gamma < 1$, and $x < \gamma$ for every element $x \in]0, 1[\subset \mathbb{R}$. A general element of Γ will then be of the form $x\gamma^n$ with $x \in \mathbb{R}_+^*$ and $n \in \mathbb{Z}$. The order is defined in the following way:

$$x\gamma^n \leq 1 \Leftrightarrow \begin{cases} x < 1 \\ \text{or} \\ x = 1 \text{ and } n \geq 0 \end{cases}$$

Beware that although $\gamma < 1$, the following inequality holds

$$0,999999 < \gamma^{100000} < 1.$$

This has to be compared (after taking logarithm) with the fact that in calculus, if ε is a negative infinitesimal element, then

$$-0,000001 < 100000\varepsilon < 0.$$

For a more precise definition, we can define Γ as the product $\mathbb{R}_+^* \times \mathbb{Z}$ with the lexicographic order, that is to say

$$(x,n) \leq (1,0) \Leftrightarrow \begin{cases} x < 1 \\ \text{or} \\ x = 1 \text{ and } n \leq 0 \end{cases}$$

With the above description, γ corresponds to $(1,-1) \in \mathbb{R}_+^* \times \mathbb{Z}$.

Now, we are ready to give the archetypal type 5 point. We will denote it by η_- but this notation is not standard.

$$\begin{array}{cccc} \eta_- : & k[T] & \to & \Gamma_0 \\ & \sum_n a_n T^n & \mapsto & \max |a_n| \gamma^n \end{array}$$

Why is this a valuation? Let $P = \sum a_i T^i$, $Q = \sum_i b_i T^i$, $PQ = \sum c_i T^i$ and let us give two reasons why $\eta_-(PQ) = \eta_-(P)\eta_-(Q)$. First, up to normalization, we can assume that $P, Q \in k^\circ[T]$ with nonzero reductions.

First reason. Let i (resp. j) be the first index such that $|a_i| = 1$ (resp. $|b_j| = 1$). It then follows from a simple calculation (which already appeared above) that the first index l for which $|c_l| = 1$ is $l = i + j$. In other words, $\eta_-(P) = \gamma^i$, $\eta_-(Q) = \gamma^j$ and $\eta_-(PQ) = \gamma^{i+j} = \eta_-(P)\eta_-(Q)$.

Second reason. If $R \in k^\circ[T]$ with nonzero reduction, then $\eta_-(R) = \gamma^{ord_T(\tilde{P})}$. where ord_T is the T-adic valuation on $\tilde{k}[T]$. □

More generally, for $c \in k$ and $r \in \mathbb{R}_+^*$, we set

$$\begin{array}{cccc} \eta_{c,r_-} : & k[T] & \to & \Gamma_0 \\ & \sum_n a_n (T-c)^n & \mapsto & \max |a_n| r^n \gamma^n \end{array}$$

Fact. If $r \notin |k^*|$ the valuation η_{c,r_-} is equivalent to the type 3 point $\eta_{c,r}$.

Proof. Let $P \in k[T]$, $\lambda \in \mathbb{R}_+^*$ and $n \in \mathbb{Z}$. We claim that

$$\eta_{c,r_-}(P) = \lambda\gamma^n \text{ if and only if } \eta_{c,r}(P) = \lambda. \tag{1}$$

$\Rightarrow$ follows from the definition. To prove $\Leftarrow$, one remarks that $|a|r^n\gamma^n = |b|r^m\gamma^m$ if and only if $|a| = |b|$ and $m = n$.

This implies that if $P, Q \in k[T]$, $\eta_{c,r}(P) < \eta_{c,r}(Q)$ if and only if $\eta_{c,r_-}(P) < \eta_{c,r_-}(Q)$. □

If $r \in \sqrt{|k^\times|}$, η_{c,r_-} defines a new valuation. It corresponds to the open ball of radius r centered in c . More precisely, if $c, c' \in k$ and $r, r' \in \mathbb{R}_+^*$, $\eta_{c,r_-} = \eta_{c',r_-}$ if and only if $B^\circ(c,r) = B^\circ(c',r')$. This is also equivalent to say that $r = r'$ and $|c - c'| < r$.

With one ultimate effort, we can define some more points. The valuation η_{c,r_-} happen to be a valuation on $k(T)$. The change of variables $T \mapsto \frac{1}{T}$ should send the point η_{c,r_-} to some new point. Let us define

$$\begin{array}{lccc} \eta_{c,r^+}: & k[T] & \to & \Gamma_0 \\ & \sum_n a_n(T-c)^n & \mapsto & \max|a_n|r^n\gamma^{-n} \end{array}$$

Beware that γ^{-1} should be thought as 1_+. This defines a new valuation of $\mathbb{A}_k^{1,\mathrm{ad}}$. And $\eta_{c,r^+} = \eta_{c',r'^+}$ if and only if $B(c,r) = B(c'r')$.

Fact. Let $\varphi : \mathbb{P}_k^{1,\mathrm{ad}} \to \mathbb{P}_k^{1,\mathrm{ad}}$ be the automorphism induced by the automorphism of $k(T)$ sending T to $\frac{1}{T}$.

(1) $\varphi(\eta_{0,1_-}) = \eta_{c,1_+}$.
(2) If $|c| = 1$, $\varphi(\eta_{c,1_-}) = \eta_{c^{-1},1_-}$

More generally

(1) If $|c| < r$, $\varphi(\eta_{c,r_-}) = \eta_{d,(r^{-1})_+}$ for any $d \in k$ such that $|d| = r^{-1}$.
(2) If $|c| = r$, $\varphi(\eta_{c,r_-}) = \eta_{c^{-1},(r^{-1})_+}$

This is coherent with the fact that if $|c| = 1$, then $B^\circ(c,1)^{-1} = B^\circ(c^{-1},1)$.

Remark 3.3. The point $\eta_{0,1^+}$ is a continuous valuation of $k\langle T\rangle$ but doesn't satisfy $\eta_{0,1^+}(k^\circ\langle T\rangle) \leq 1$. If one sets $C := \{\sum_n a_nT^n \mid |a_0| \leq 1 \text{ and } |a_i| < 1,\ i > 0\} \subset k\langle T\rangle$, then $(k\langle T\rangle, C)$ is an affinoid ring, and $\mathrm{Spa}(k\langle T\rangle, C) = \mathbb{B} \cup \{\eta_{0,1^+}\}$.

Remark 3.4. For all c' such that $|c'-c| \leq r$, $\eta_{c',r^-} \in \overline{\{\eta_{c,r}\}}$, and $\eta_{c,r^+} \in \overline{\{\eta_{c,r}\}}$ and this describes all the points of $\overline{\{\eta_{c,r}\}}$. More precisely, inside $\mathbb{A}_k^{1,\mathrm{ad}}$, $\overline{\{\eta_{c,r}\}}$ is isomorphic to $\mathbb{P}^1_{\tilde{k}}$. Beware that inside $\mathbb{B}$, $\overline{\{\eta_{0,1}\}}$ is isomorphic to $\mathbb{A}^1_{\tilde{k}}$.

3.3. Proof of the classification.

Proposition 3.5. *When $k = k^{alg}$, $\mathbb{A}_k^{1,ad}$ consists precisely of the points of type* 1 *to* 5.

Proof. Let $v : k[T] \to \Gamma$ be some valuation of $\mathbb{A}_k^{1,\mathrm{ad}}$.

Case 0. Let us assume that $\mathrm{supp}(v) \neq \{0\}$. This must be some nonzero prime ideal of $k[T]$, hence of the form $(T-c)$ for some $c \in k$. So v must factorize as $v : k[T] \to k \to \Gamma_0$ and thus must be a type 1 point.

So we can assume that $\mathrm{supp}(v) \neq \{0\}$, hence we can assume that v actually comes from a valuation $v : k(T) \to \Gamma_0$ extending the norm of k. So we just have to understand the possible ways to extend[4] a valuation to $k(T)$.

Case 1. Let us assume that $\Gamma = |k^*|$ and $\widetilde{k(T)} = \tilde{k}$. This means v is a type 4 point.

Case 2. Let us assume that $\Gamma = |k^*|$ and $\tilde{k} \subsetneq \widetilde{k(T)}$. Then there must be some $a \in k$ and $b \in k^*$ such that $v(\frac{T-a}{b}) = 1$ and $\widetilde{(\frac{T-a}{b})} \notin \tilde{k}$, hence is a transcendental element. Up to a change of variables, we can assume that $v(T) = 1$ and $\tilde{T}$ is transcendental. Let us consider some $P = \sum_i a_iT^i \in k^\circ[T] \setminus k^{\circ\circ}[T]$. Then $v(P) \leq 1$ and the reduction of P in $\widetilde{k(T)}$ is $\sum \tilde{a}_i\tilde{T}^i$ which is nonzero because $\tilde{T}$ is transcendental and one of the $\tilde{a}_i$'s is nonzero by assumption. So $v(P) = \eta(P)$. Hence v is a type 2 point.

[4] Let us mention that any good reference about valuation theory, for instance the section [Bou98, VI Chapter 10] entitled *Prolongements d'une valuation à une extension transcendante* contains the tedious case disjunction which follows.

Case 3. Let us assume that $|k^*| \subsetneq \Gamma$. So there exists $c \in k$ such that $\xi := v(T-c) \notin |k^*|$. A calculation already made before implies that

$$v(\sum a_i (T-c)^i) = \max_i |a_i| \xi^i. \tag{2}$$

Moreover, this maximum is attained on one single index i.

Case 3.1. Let us assume that Γ is of rank one. Then there exists a (unique) embedding $\Gamma \subset \mathbb{R}_+^*$ which preserves our initial embedding $|k| \subset \mathbb{R}_+^*$. So we can assume that $\Gamma \subset \mathbb{R}_+^*$ and $\xi \in \mathbb{R}_+^* \setminus |k^*|$. Then $v = \eta_{c,\xi}$.

Case 3.2. Let us assume that Γ is not of rank one. According to the formula (2), one sees that $\Gamma \simeq |k^*|\xi^{\mathbb{Z}}$. Remind that the only condition we put on v to be in $\mathbb{A}_k^{1,\mathrm{ad}}$ is that the convex subgroup generated by $|k^*|$ must be Γ itself, in other words that there are no elements in Γ which are infinitesimally bigger than $|k^*|$. The only possibility is that Γ should be an ordered subgroup of $|k^*| \times \mathbb{Z} \subset \mathbb{R}_+^* \times \mathbb{Z}$ with the lexicographic order, or in other words that Γ should be an ordered subgroup of $\mathbb{R}_+^* \gamma^{\mathbb{Z}}$, the value group we have considered to define type 5 points. So ξ should be sent to some $r\gamma^n$ with $r \in \mathbb{R}_+^*$ and $n \in \mathbb{Z}$.

Claim. $r \in |k^*|$.

If not, $r \in \mathbb{R}_+^* \setminus |k^*|$. We subclaim that for $a \in k^*$ and $n \in \mathbb{Z}$

$$|a|\xi^n < 1 \Leftrightarrow |a| r^n \gamma^n < 1 \Leftrightarrow |a| r^n < 1.$$

Only the last equivalence has to be proved. $\Leftarrow$ is true because γ is infinitesimally closed to 1. For $\Rightarrow$, if $|a|r^n\gamma^n < 1$, it follows that $|a|r^n \leq 1$, because γ is still infinitesimally closed to 1. But if $|a|r^n = 1$ then necessarily $|a| = 1$ and $n = 0$ because we have assumed that $r \in \mathbb{R}_+^* \setminus |k^*|$. But this would contradict the fact that $|a|r^n\gamma^n < 1$. This proves the subclaim. This subclaim would imply that $v \simeq \eta_{c,r}$ which would be a contradiction.

So, $r \in |k^*|$, and up to a change of variables, we can assume that $v(T) = \gamma^{\pm 1}$, i.e. that v is of type 5.

This ends the proof since there are no more cases. □

3.4. **Local rings.** Let us describe the local rings in $\mathbb{A}_k^{1,\mathrm{ad}}$.

Type 1. The local ring of the origin is

$$\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{ad}},0} = \varinjlim_{r>0} k\langle r^{-1}T \rangle.$$

Type 2. It is enough to consider η, the other local rings being isomorphic to it. A basis of neighborhood of η is given by the affinoid domains defined by the inequalities

$$\{|T - a_i| = 1,\ i = 1 \dots n\}$$

where $a_1, \dots, a_n \in k^\circ$. In fact only the residue classes $\tilde{a_i}$ matter. So

$$\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{ad}},\eta} = \varinjlim_{\substack{n \in \mathbb{N} \\ \tilde{a_1}, \dots, \tilde{a_n} \in \tilde{k}}} \mathcal{O}(\{|T - a_i| = 1,\ i = 1 \dots n\}).$$

These affinoid rings have very natural descriptions in terms of Laurent series (see for instance [FvdP04, 2.2.6]):

$$\mathcal{O}(\{|T - a_i| = 1,\ i = 1 \dots n\}) \simeq \{\sum_{\nu \in \mathbb{Z}^n} a_\nu (T-a_1)^{\nu_1} \cdots (T-a_n)^{\nu_n} \mid |a_\nu| \xrightarrow{|\nu| \to +\infty} 0\}.$$

Type 3. Let $r \in \mathbb{R}_+^* \setminus |k^*|$. A basis of neighborhood of η_r is given by the affinoid domains $\{s_1 \leq |T| \leq s_2\}$ where $s_i \in |k^*|$ and $s_1 < r < s_2$. So

$$\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{ad}},\eta_r} = \varinjlim_{\substack{s_1,s_2 \in |k^*| \\ s_1 < r < s_2}} \mathcal{O}(\{s_1 < |T| < s_2\}).$$

Type 4. If $x \in \mathbb{A}_k^{1,\mathrm{ad}}$ corresponds to a decreasing sequence of balls $\{B_i\}$ with empty intersection, then

$$\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{ad}},x} = \varinjlim_i \mathcal{O}(B_i).$$

Type 5. A basis of neighborhood of $\eta_{0,1_-}$ is given by the affinoid domains $\{s \leq |T|,\ |T - a_i| = 1,\ i = 1 \dots n\}$ where $|a_i| = 1$ and $s < 1$. Hence

$$\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{ad}},\eta_{0,1_-}} = \varinjlim_{\substack{n \in \mathbb{N},\ s<1 \\ \tilde{a_1},\dots,\tilde{a_n} \in \tilde{k}^*}} \mathcal{O}(\{s \leq\ |T|,\ |T - a_i| = 1,\ i = 1 \dots n\}).$$

Let us remark that

$$\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{ad}},\eta_{0,1_+}} = \varinjlim_{\substack{n \in \mathbb{N},\ s>1 \\ \tilde{a_1},\dots,\tilde{a_n} \in \tilde{k}^*}} \mathcal{O}(\{s \geq\ |T|,\ |T - a_i| = 1,\ i = 1 \dots n\}).$$

There is a natural inclusion $\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{ad}},\eta_{0,1_-}} \hookrightarrow \mathcal{O}_{\mathbb{A}_k^{1,\mathrm{ad}},\eta_{0,1}}$. One last remark: except for type 1 points, these local rings are fields.

3.5. **Why not in higher dimensions.** According to the above classification, up to isomorphism, there are few type of points in $\mathbb{A}_k^{1,\mathrm{ad}}$. We have shown that if x, x' are points of the same type, the type being $1, 2$ or 5, there exists an automorphism φ of $\mathbb{P}_k^{1,\mathrm{ad}}$ such that $\varphi(x) = x'$. Two type 3 points $\eta_{c,r}$, $\eta_{c',r'}$ have the same completed residue field if and only if $\frac{r}{r'} \in \||k^*|$ or $rr' \in |k^*|$.

A general comment: to any point $x\ in (\mathbb{A}_k^n)^{\mathrm{ad}}$, one can associate three invariants to $k(x)$, namely.

$$\begin{aligned} a = &\quad \mathrm{tr.deg.}(\widetilde{k(x)}/k) \\ b = &\quad \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} (|k(x)^*|/|k^*|) \\ c = &\quad \mathrm{rank}(|k(x)^*|) - \mathrm{rank}(|k^*|) \end{aligned}$$

These invariants satisfy the following relations: $b \leq c$ and $a + b \leq n$. In the case of the line each type of points corresponds to one of the five possibilities for these a, b, c.

In higher dimension, we can of course find some familiar points. For instance, if we define Γ as an ordered group, containing $\mathbb{R}_+*$, and adding some generic elements γ, γ' such that $r\gamma^n\gamma'^m < 1$ if and only if $r < 1$ or ($r = 1$ and $n > 1$) or ($r = 1$, $n = 1$, and $m > 0$). Then $\sum a_{i,j}T^iU^j \mapsto \max |a_{i,j}|\gamma^i\gamma'^j \in \Gamma$ defines a new points. But even for $\mathbb{A}_k^{2,\mathrm{ad}}$ more complicated phenomena appear. Let us list some of them.

(1) The residual extension $\widetilde{k(x)}/\tilde{k}$ might not be purely transcendental and functions fields of higher genus over $\tilde{k}$ appear.
(2) We might take $\Gamma \subset \mathbb{R}_+^* \times \gamma^{\mathbb{Z}+\sqrt{2}\mathbb{Z}}$ with γ some kind of 1_- and define a valuation $\sum a_{i,j}T^iU^j \mapsto \max |a_{i,j}|\gamma^i\gamma'^{\sqrt{2}j} \in \Gamma$.

4. Finite type, proper, étale … morphisms

For simplicity we assume that R is a Tate ring.

Definition 4.1. A morphism $f : (R, R^+) \to (S, S^+)$ between affinoid rings is a quotient map if f is surjective continuous and open, and S^+ is the integral closure of $f(R^+)$ in S.
If (R, R^+) is an affinoid ring with R Tate, we set

$$(R, R^+)\langle T_1 \dots T_n\rangle = (R\langle T_1 \dots T_n\rangle, R^+\langle T_1 \dots T_n\rangle)$$

which is an affinoid ring[5].

Definition 4.2. A morphism of affinoid rings $(R, R^+) \to (S, S^+)$ is of topologically finite type if it factorizes as $(R, R^+) \to (R, R^+)\langle T_1 \dots T_n\rangle \xrightarrow{\pi} (S, S^+)$ where π is a quotient map.

Definition 4.3. Let $f : X \to Y$ be a morphism of adic spaces. It is

1. locally of weakly finite type if for all $x \in X$ there exists U, V, open affinoid subspaces of X, Y, such that $x \in U$, $f(U) \subseteq V$ and such that the morphism of f-adic ring $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of topologically finite type.
2. locally of finite type if for all $x \in X$ there exists U, V, open affinoid subspaces of X, Y, such that $x \in U$ and $f(U) \subseteq V$ and such that the affinoid ring morphism $(\mathcal{O}_Y(V), \mathcal{O}^+_Y(V)) \to (\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ is of topologically finite type.
3. f is of finite type if it is quasi-compact and locally of finite type.

Proposition 4.4. *If $f : X \to Z$ and $g : Y \to Z$ are morphisms of adic spaces with f locally of finite type, one can define the fiber product of f and g.*

Definition 4.5. If $f : X \to Y$ is a morphism of adic spaces locally of finite type [6], then

1. f is separated if $\Delta(X)$ is closed in $X \times_Y X$ where Δ is the diagonal morphism.
2. f is universally closed if it is locally of weakly finite type, and for all adic morphism [7] $Y' \to Y$, $X \times_Y Y' \to Y'$ is closed.
3. f is proper if it is of finite type, separated and universally closed.

In [Hub96, 1.3] a kind of valuative criterion for properness is proved.
$\mathbb{B}$ =Spa$(k\langle T\rangle, k^\circ\langle T\rangle)$ is not proper, however, if $C = \{\sum_n a_n T^n \mid |a_0| \leq 1 \text{ and } |a_i| < 1,\ i > 0\}$, then Spa$(k\langle T\rangle, C)$ is proper.
This notion of properness is related to the notion of properness of rigid k-spaces, see [Hub96, Rem 1.3.19].

Definition 4.6.

1. A morphism $f : (R, R^+) \to (S, S^+)$ of affinoid rings is finite if it is of topologically finite type, $R \to S$ is finite, and S^+ is the integral closure of $f(R^+)$.

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[5] One has to check that $R^+\langle T_1 \dots T_n\rangle$ is integrally closed in $R\langle T_1 \dots T_n\rangle$.

[6] Huber needs only f to be locally of $^+$weakly finite type which is more general than locally of finite type.

[7] see p.46 of [Hub96] for the definition of an adic morphism

(2) A morphism of adic spaces $f : X \to Y$ is finite if for all $y \in Y$, there exists an affinoid neighborhood U such that $f^{-1}(U) = V$ is affinoid and $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite.
(3) f is locally quasi-finite if $\forall y \in Y$, $f^{-1}(y)$ is discrete.
(4) f is quasi-finite if f is quasi-compact and locally quasi-finite.

Definition 4.7. If (R, R^+) is an affinoid ring, and I an ideal of R, we denote by $(R, R^+)/I$ the affinoid ring $(R/I, (R^+/(I \cap R^+))^c)$ where $(R^+/(I \cap R^+))^c$ is the integral closure of $R^+/(I \cap R^+)$ in R/I.[8]

Definition 4.8.

(1) A morphism $f : X \to Y$ which is locally of finite type is called unramified (resp. smooth, resp. étale) if for all affinoid ring (R, R^+), and all ideal I of R such that $I^2 = \{0\}$, and $g : \mathrm{Spa}(R, R^+) \to Y$, $\mathrm{Hom}_Y(\mathrm{Spa}(R, R^+), X) \to \mathrm{Hom}_Y(\mathrm{Spa}((R, R^+)/I), X)$ is injective (resp. surjective, resp. bijective).
(2) A morphism $f : X \to Y$ is said to be unramified (resp. smooth, resp. étale) at $x \in X$ if there exist U, V open subsets of X, Y such that $x \in U$ and $f(U) \subseteq V$ and $f_{|U} : U \to V$ is unramified (resp. smooth, resp. étale).

One can define sheaf of differentials $\Omega_{X|Y}$ (which is an $\mathcal{O}_X$-module), when $f : X \to Y$ is locally of finite type, such that

Proposition 4.9.

(1) *f is unramified if and only if $\Omega_{X|Y} = 0$.*
(2) *If f is smooth, $\Omega_{X|Y}$ is a locally free $\mathcal{O}_X$-module.*
(3) *If (R, R^+) is an affinoid ring with R Tate, $Y = Spa(R, R^+)$, then $f : X \to Y$ is smooth if and only if for all $x \in X$ there exists a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\ g\ } & Spa((R,R^+)\langle T_1, \ldots, T_n\rangle) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle h} & \\ Spa(R, R^+) & & \end{array}$$

where U is an open neighborhood of x, h is the natural morphism and g is étale.

Proposition 4.10. *Let $f : X \to Y = Spa(R, R^+)$ be a morphism of affinoid adic spaces, with R Tate. The following are equivalent:*

(1) *f is étale.*
(2) *There exists $n \in \mathbb{N}$, $f_1 \ldots f_n \in R\langle T_1 \ldots T_n\rangle$ such that if I is the ideal $(f_1, \ldots, f_n)$ then $det(\frac{\partial f_i}{\partial T_j})_{1 \le i,j \le n}$ is invertible in $R\langle T_1 \ldots T_n\rangle/I$, and X is Y-isomorphic to $Spa((R, R^+)\langle T_1 \ldots T_n\rangle/I)$.*
(3) *There exists $n \in \mathbb{N}$, $f_1 \ldots f_n \in R[T_1 \ldots T_n]$ such that if I is the ideal $(f_1, \ldots, f_n)$ then $det(\frac{\partial f_i}{\partial T_j})_{1 \le i,j \le n}$ is invertible in $R\langle T_1 \ldots T_n\rangle/I$, and X is Y-isomorphic to $Spa((R, R^+)\langle T_1 \ldots T_n\rangle/I)$.*

[8]that we equip with the quotient topology.

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