Huber Continuous valuations

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Préliminaires

0.1 Valuation ring

Définition 0.1 *cf [8, p.71] An integer ring R is a valuation ring if, noting K for its field of fraction, $\forall x \in K$, $x \in R$ or $x^{-1} \in R$.*

cf aussi [1, p122]

Proposition 0.1 *If R is a valuation ring, R is a local ring. The ideals of R are totally ordered by inclusion.*

D : R is local iff the non-invertible elements are stable by addition. If x and y are not invertible, if x or y is 0, then OK, otherwise, $\frac{x}{y}$ say is in R.Then $x + y = y + y\frac{x}{y} = y(1 + \frac{x}{y})$, and since y isn't invertible, $x + y$ neither.
Let I and J be two ideals of R. Suppose $I \nsubseteq J$ and $J \nsubseteq I$. let then $i \in I\backslash J$ and $j \in J\backslash I$. By symetry, suppose that $\frac{i}{j} \in R$. Then $i = j\frac{i}{j} \in J$, which is absurd. □

Actually, this also proves that R is a local ring : define $\mathfrak{m}$ as the union of all the proper ideals of R. It is a proper ideal beceause they are totally orderd by inclusion, and maximal.

Proposition 0.2 *Let K be a field, $v : K \to \Gamma_0$ such that*
$v(x) = 0 iff x = 0$
$v(xy) = v(x)v(y)$.
Then v is a valuation iff $\forall x$ such that $v(x) \leqslant 1$, $v(1+x) \leqslant 1$.

D : $\Rightarrow$ $v(1+x) \leqslant \max(v(1), v(x)) = 1$.
$\Leftarrow$: let x and $y \in K$ not zero (otherwise it is easy). Suppose $v(x) \geqslant v(y)$. Hence $v(\frac{y}{x}) \leqslant 1$. Then $v(x+y) = v(x)v(1+\frac{y}{x}) \leqslant v(x)$. □

Définition 0.2 *A valuation on A is $v : A \to \Gamma_0$ such that $v(ab) = v(a)v(b)$ and*

1. *[2, VI, §3] or [8, p. 75] , $v(x+y) \geqslant \min(v(x), v(y))$ and the law on Γ_0 is that 0 is ∞ , i.e. greatest element (we should note in fact Γ_∞*
2. *[5] $v(x+y) \leqslant \max(v(x), v(y))$, and 0 is the lowest element.*

Putting $w := \frac{1}{v}$, i.e. $w(a) = v(a)^{-1}$ if $v(a) \neq 0$, and $w(a) = 0$ if $v(a) = 0$ (in fact $0 \leftrightarrow \infty$) , gives a bijection between valuation of hte types (i) and (ii). We will always take definition (ii).

0.2 product of valuation

If $v_i : A \to Gamma_{i0}$ are two valuations ($i = 1, 2$), then
$v : A \to \Gamma_1 \times \Gamma_{20}$ (with the lexicographic order) is not a valuation in general. Indeed if one can find a and b such that $v_1(a) < v_1(b)$ and $v_2(b) < v_2(a)$, then we would have $v(a+b) = (v_1(b), v_2(a)) > \max(v(a), v(b)) = v(b) = (v_1(b), v_2(b))$.
Exemples :
* $A = \mathbb{Z}$, $v_2 the 2-adic$ valuation, and v_3 the 3-adic one, then $v(2+3) = (0,0), v(2) = (-1,0), v(3) = (0,-1)$.

*$A = k\{T\}$. $v_1 = \eta_{B(0,1)}$ and $v_2 = \eta_{B(0,\frac{1}{p}}$.
Then $v(T^2+p) = (1, \frac{1}{p})$, $v(T^2) = (1, \frac{1}{p^2})$, $v(p) = (\frac{1}{p}, \frac{1}{p})$.

0.3 Completion of a topological ring

Let A be a topological ring. A sequence in A is said to be a Cauchy sequence if for every 0-neighborhood V there exists N such that $n, m \geqslant N$ implies $x_n - x_m \in V$.
If $x_n \to l$, then x_n is Cauchy. Indeed let V be the 0 neigborhood, then there exists a 0-neigborhood W such that $W - W \subseteq V$. Then for n big enough $x_n \in l + W$, and $x_n - x_m \in W - W \subseteq V$.

A_{cauchy} is a group for $+$. If x_n and y_n are Cauchy, let V be a 0-neigborhood. Let W be another one such that $W + W \subseteq V$. Then for n big enough, $x_n - x_m \in W$ and the same thing for y, so that $(x+y)_n - (x+y)_m \in W + W \subseteq V$.

x_n Cauchy $\Rightarrow$ x_n bounded. Let V be a 0-neighborhood, and W st $W + W \subseteq V$ (in particular $W \subseteq V$). Let X_1, X_2 two neigborhoods s t $X_1.X_2 \subseteq W$. $\exists N$ s t $n \geqslant N \Rightarrow (x_n - x_N) \in X_1$.
for $i = 0 \dots N-1$ $\exists W_i$ s t $x_i W_i \subseteq W$.
Define $Z = W_0 \cap \dots \cap W_N \cap X_2$. Then for $i = 0 \dots N-1$ $x_i Z \subseteq x_i W_i \subseteq W \subseteq V$.
For $i \geqslant N$, $x_i Z \subseteq (x_i - x_n)Z + (x_N)Z \subseteq X_1.X_2 + x_N W_N \subseteq W + W \subseteq V$.

x_n and y_n Cauchy $\Rightarrow$ $(xy)_n$ are Cauchy Let V. $\exists W$ s t $W + W \subseteq V$.
$\exists X_x and X_y$ s t $x_n.X_x$ and $y_n.X_y \in W$.
$X = X_x \cap X_y$.
Then for $n, m \gg 0$ $y_m - y_n$ and $x_m - x_n \in X$ then $x_m y_m - x_n y_n \in W + W \subseteq V$.

the 0 sequence form an ideal of A_{cauchy} Let x_n be a 0 sequence, and y_n a Cauchy one, so that it is bounded.
Let V. As y_n is bounded there exists W such that $y_n.W \subseteq V$. then for $n \gg 0$ $x_n \in W$, and $(xy)_n \in V$.

– Define $hatA = A_{cauchy}/zero - seq$.

0.4 Ordered groups

Γ is an abelian ordered group.

Proposition 0.3 *$a \geqslant b$ iff $a^{-1} \geqslant b^{-1}$.*

D : $a \geqslant b \Rightarrow a(a^{-1}b^{-1}) \geqslant b(a^{-1}b^{-1})$, i.e. $b^{-1} \geqslant a^{-1}$. Applying this with a^{-1} and b^{-1} gives the other inequality. □.

Définition 0.3 *A subset X of Γ is convex if $\forall\ x, y, z \in \Gamma$ with $x \leqslant y \leqslant z$ and $x, z \in X$ then $y \in X$.*

The convex sets are stable by intersection ; if X_i are convex sets that all contain 1, then $\cup X_i$ is convex. If X is convex, $X^{-1} = \{x^{-1} \mid x \in X\}$ is convex .

Définition 0.4 *If $A \subseteq \Gamma$, define $A_{conv} = \{x \in \Gamma \mid \exists a, b \in A witha \leqslant x \leqslant b\}$. This is the smallest convex set that contains A.*

Proposition 0.4 *If $H \subseteq \Gamma$ is a subgroup , H_{conv} is a subgroup.*

Indeed if $a \leqslant x \leqslant b$, and $a' \leqslant x' \leqslant b'$ then $aa' \leqslant xx' \leqslant bb'$, and $a^{-1} \geqslant x^{-1} \geqslant b^{-1}$. □

Proposition 0.5 *If $X \subseteq \Gamma$ is a convex subset that contains 1 , then $< X >$ (the subgroup generated by X) is convex.*

D : Let's show that $X.X$ is convex : let $a, b, c, d \in X$ and $ab \leqslant x \leqslant cd$. Put $\{a, b, c, d\} = \{\alpha, \beta, \gamma, \delta\}$, s t $\alpha \leqslant \beta, \gamma, \delta$. Then $\alpha\beta \leqslant ab \leqslant x \leqslant cd \leqslant \gamma\delta$, so that we can assume that $a \leqslant b \leqslant c \leqslant d$. Then $ab \leqslant cb \leqslant cd$.
If $ab \leqslant x \leqslant cb$, $a \leqslant xb^{-1} \leqslant c$, and $xb^{-1} \in X$ since it is convex, so since $b \in X$, $x = xb^{-1}.b \in X.X$.
Otherwise $cb \leqslant x \leqslant cd$ and $b \leqslant xc^{-1} \leqslant d$ and $x = x^{c-1}.c \in X.X$.
Define $X' = X \cap X^{-1}$ which is convex since $1 \in X$ and X^{-1} , then $< X >= \cap_{n>0} X'^n$ is then convex. □

The hypothesis $1 \in X$ is necessary as shows the exemple $X = \{2\} \subseteq (\mathbb{Z}, +, \leqslant)$, where $< 2 >= 2\mathbb{Z}$ is not convex.

Corollaire 0.1 *If $X \subseteq \Gamma$ $< (X \cup \{1\})_{conv} >=< X >_{conv}$*

Indeed $< (X \cup \{1\})_{conv} >\supseteq< X >$ and is convex so $< (X \cup \{1\})_{conv} >\supseteq< X >_{conv}$. And $< X >_{conv} \supset X \cup \{1\}$ and is a group , so $< X >_{conv}\supseteq< (X \cup \{1\})_{conv} >$. □

Define Conv(Γ) as the set of convex subgroups of Γ. Then it is totally orderd for inclusion. It has the lower and upper bound properties (take $\cap$ and $\cup$).
Call a convex subgroup $< g >_{conv}$ a principal subgroup. Note that a convex subgroup isn't neceseraly principal.
For instance take $\Gamma = (\mathbb{Z}, +, \leqslant)^{(\mathbb{Q})}$, that is the sequences indexed by $\mathbb{Q}$ almost everywhere zero, and ordered by the lexicographic order. Then $H_{\sqrt{2}} = \{x \in \Gamma | supp(x) \subseteq]-\infty, \sqrt{2}]\}$ is a convex subgroup , which is not principal, since any principal subgroup is of the form $H_a = \{x \in \Gamma | supp(x) \subseteq]-\infty, a]\}$ for $a \in \mathbb{Q}$.
In fact for any $b \in \mathbb{R}$, $H^b = \{x \in \Gamma | supp(x) \subseteq] - \infty, b[\}$ is also a non principal subgroup.

Définition 0.5 *Let $H \subseteq \Gamma$ a subgroup, then $\gamma \in \Gamma$ is cofinal in H if $\forall h \in H, \exists n \in \mathbb{N}$ s t $\gamma^n < h$.*

Proposition 0.6 *g is cofinal in H iff $< g >_{conv}\supseteq H$ and $g < 1$.*

$\Rightarrow$: Since $\exists n$ s t $1 > g^n$, we have $1 > g$. let $h \in H$. If necessary, let's take h^{-1}, so that $h \leqslant 1$. Then there exists a $n \geqslant 0$ s t $1 \geqslant h \geqslant g^n, \Rightarrow h \in< g >_{conv}$.
$\Leftrightarrow$: let $h \in H$, here again, taking h^{-1} if necessary, we can assume that $1 \geqslant h$. Then there exists a $n \in \mathbb{N}$ s t $g^n \leqslant h \leqslant 1$, and then $g^{n+1} < h$, so g is cofinal in H. □.

Proposition 0.7 *Let (H_i) be a increasing family of subgroups s t g is cofinal in each H_i. Then g is cofinal in $H = \cup H_i$.*

D : if $h \in H$ then $h \in H_i$ for one i and then g being cofinal for H_i $\exists n$ s t $g^n < h$. □

Corollaire 0.2 *Let $g < 1$. Then there exists a bigger convex subgroup H of Γ s t g is cofinal in H. In fact $H =< g >_{conv}$*

D : The family $\mathcal{F}$ of convex subgroups G for which g is cofinal is non empty $\{1\}$ works, then $H = \cup_{G \in \mathcal{F}} G$ is a convex subgroup, and g is cofinal for H according to the previous proposition. Clearly it is maximal for this property.
Now, g is cofinal in $< g >_{conv}$, indeed $x \in < g >_{conv}$ implies there exists $n \geqslant 0$ such that $g^n \leqslant x$ so $g^{n+1} < x$ and so $H \supseteq < g >_{conv}$. Now if $h \in H$, takin h^{-1} if necessary we can assume $h \leqslant 1$. Then $\exists n > 0$ s t $g^n < h \leqslant 1$ and $h \in < g >_{conv}$. □

Remarque 1 *Let $X \subseteq \Gamma_{<1} = \{g \in \Gamma \mid g < 1\}$. Define*
$Conv(X) = \{H$ convex subgroup such that $\forall x \in X$, x is cofinal in $H\}$. Since it stable by $\supseteq$, and nonempty ($\{1\} \in Conv(X)$) we can (taking its lower bound, i.e. intersection) see it has a subgroup : the smallest such that...Then from what we have done, $Conv(X) = \cap_{x \in X} < x >_{conv}$.

1 F-adic rings

Proposition 1.1 $A^{\circ\circ} \subseteq A^{\circ}$

D : Let $a \in A^{\circ\circ}$, and V a 0 neigborhood. There exists W a 0 neigborhood such that $W.W \subseteq V$. There exists N such that $n \geqslant N$ implies $a^n \in W$. For each $i = 0 \ldots N-1$ there exsists W_i a 0 neigborhood s t $a^i W_i \subseteq V$. Then if $U = W_0 \cap \ldots \cap W_{N-1} \cap W$ then for each i $a^i U \subseteq V$. □

An adic ring is bounded . Indeed, if I is an ideal of definition of A , then if V is a 0 neigborhood, there exists a n s t $I^n \subseteq V$ and $I^n A = I^n \subseteq V$.
For S and T two susbsets of A, le $S.T$ be subgroup of $(A, +)$ generated by the elements st , $s \in S$ and $t \in T$.

Définition 1.1
1. *A topological ring A is f-adic if there exists a subset U and a finite subset of U, T such that $\{U^n \| n \in \mathbb{N}\}$ is a fundamental system of 0 neigborhood, and $T.U = U^2 \subseteq U$.*
2. *A is called a Tate ring if it is f-adic and has a topologically nilpotent unit.*

A ring of definition of a f-adic ring is an open subring A_0 of A which is adic.

Proposition 1.2 (Prop 1) *Let A ba a f-adic ring. Then*
1. *A has a ring of definition.*
2. *A subring A_0 is a ring of definition iff it is open and bounded.*
3. *Every ring of definition of A has a finitely generated ideal of definition.*

It is then clear that a topological ring A is f-adic iff it has a an open subring A_0, which is adic for a finitely generated ideal I (since in this case A is clearly f-adic).

D : Let W be the subgroup of A generated by U. Since $U^2 \subseteq U$, we can conclude that $W^2 \subseteq W$. Let $B = \mathbb{Z} + W$. Then B is a subgroup of A for its additive law. It is also stable by multiplication : $(n+w).(m+w') = nm + mw + nw' + ww' \in B$ ($W^2 \subseteq W$). B is then a subring. It is open since it contains U which is an 0 neigborhood , and a subgroup of a topological group is open iff it contains a 0 neigborhood. For $n \geqslant 2$, $B.U^n = U^n$, because $\mathbb{Z}.U^n = U^n$, and $W.U^n = U^{n+1}$, and the fact that $U^2 \subseteq U$ implies that $U^{n+1} \subseteq U^n$. Hence the U^n being a fundamental system of neigborhoods of 0 implies that B is open.
Hence we can introduce A_0 an open and bounded subring of A.
For $n \in \mathbb{N}$ define the finite set $T(n) = \{t_1.t_2, \ldots t_n \mid t_i \in T\}$. Since $T \subseteq U$ and $T.U = U^2$,

$T(n) \subseteq U^n$. In particular since the U^n form a fundamental system of neigborhood and A_0 is open, $\exists k$ s t $T(k) \subseteq A_0$. Put then $I = t(k).A_0$. Let's show that I^n (seen here as an ideal of A_0) is a fundamental system of neigborood of 0 (in A, or A_0, it is equivalent since A_0 is open).
First, there exists a m s t $U^m \subseteq A_0$, and then for $n \in \mathbb{N}$, one easily sees that $I^n = T(nk)A_0 \supseteq T(nk)U^m = U^{nk+m}$, so I^n is a 0 neigborhhod.
Let V be a 0 neigborhood. Then there exists m s t $U^l A_0 \subseteq V$ because A_0 is bounded. But now $I^m = T(mk)A_0 \subseteq U^{mk}A_0 \subseteq U^m A_0 \subseteq V$. Hence A_0 is a ring of definition for I, and I is of finite type, which proves (i) and (ii).
Now if A_0 is a ring of definition of A, as noted previously , A_0 is bounded (in A_0, so in A too) , since it is adic. So by what we have done, it has a finitely generated ideal of definition.□

Now then, for A a f-adic ring , we will consider it coming with a couple (A_0, I) , with A_0 a ring of definition and I an ideal of definition. Then the I^n form a fundamental system of neigborhood of 0.

Lemme 1.1 *Let A be a f-adic ring, S and T bounded subsets. Then $S.T$ is bounded.*

D : let (A_0, I) a ring of definition, and I^n a 0 neigborhood. $\exists m$ s t $SI^m \subseteq I^n$. $\exists p$ st $TI^p \subseteq I^m$. Then, if $s, t \in S \times T$, and $a \in I^p$, $ta \in I^m$ so $sta \in I^n$. Since I^n is a subgroup, one then conclude that $(S.T)I^p \subseteq I^n$. □

Corollaire 1.1 *Let A be a f-adic ring.*

1. *If A_0 and A_1 are rings of definition, then $A_0.A_1$ and $A_0 \cup A_1$ also.*
2. *If B is a bounded subring, and C an open subring with $B \subseteq C$ subseteq$A$, there exists A_0 a ring of definition with $B \subseteq A_0 \subseteq C$*
3. *A° is a subring , and it is the union of all rings of definition.*

D :
(i) the second point of the previous proposition shows that A_0 and A_1 are open and bounded. Then $A_0 \cup A_1$ is also open, and bounded . Then $A_0.A_1$ is also open (it contains A_0) , and bounded according to the lemma. So the second point of the proposition shows that there are ring of definition.
(ii) Let A_1 be a ring of definition. Then $B.A_1$ is a subring, bounded (previous lemma) , and open (contains A_0) , so is a ring of definition. Then $A_0 = A_1 \cup C$ is an open bounded subring so is a ring of definition, and $B \subseteq A_0 \subseteq C$.
(iii) Let (A_0, I) be a ring of definition for A. First 0 and $1 \in A^\circ$.
Let now $a, b \in A^\circ$, and I^n be a 0 neigborhood. There exists m s t $\{a^k, k \in \mathbb{N}\}I^m \subseteq I^n$ and $\{b^k\}I^m \subseteq I^n$. Then for $r, s \in \mathbb{N}$, $a^r b^s I^{2m} = a^r(b^s I^m)I^m \subseteq a^r I^{n+m} \subseteq a^r I^m \subseteq I^n$. Since $(a+b)^k = \sum \binom{k}{l} a^l b^{k-l}$, one has $(a+b)^k I^{2m} \subseteq I^n$, and $(ab)^k I^{2m} \subseteq I^n$, i.e. $a+b$ and $ab \in A^\circ$. So A° is a subring.
Now if X is bounded, then $X \subseteq A^\circ$, in particular for any A_0 ring of definition, $A_0 \subseteq A^\circ$. On the other hand, $\mathbb{Z}$ is bounded (this is a consequence of the fact that some ring of definition exist, that they are bounded and contain $\mathbb{Z}$, more simply because $\mathbb{Z}I^n \subseteq I^n$). Let now $x \in A^\circ$. Then by definition $\{x^n\}$ is bounded, so $B = \mathbb{Z}.\{x^n\} = \mathbb{Z}[x]$ is a bounded subring. $B \subseteq A$ which is open (!) , so with (ii) there exists a ring of definition A_0 with $B \subseteq A_0$, and then $x \in A_0$ So $A^\circ = \cup_{A_0} A_0$.□

Proposition 1.3 *If A is f-adic $A^{\circ\circ}$ is a subring (except it doen't contain 1)*

same proof

Corollaire 1.2

1. *An adic ring is f adic iff it has a finitely generated ideal of definition.*
2. *A f adic ring is adic iff it is bounded*
3. *Let A be a topological ring and B an open subring. Then A is f adic iff B is.*

D :

(i) $\Rightarrow$ is a consequence of (ii) of the prop.
$\Leftarrow$: already seen.
(ii) Let A be f-adic. If A is adic it is bounded (this is true without the assumption f-adic). Conversly , if A is bounded, then (ii) of the proposition , since A is bounded and open in A f-adic, it is a ring of definition, hence is adic.
(iii) If B is f-adic, one can find (B_0, I) ring of definition for B, and since B is open , (B_0, I) is also a ring of definition for A hence A is f adic. Conversly, if A is f-adic, and B an open subring. Then $\mathbb{Z}$ is a bounded subring, and $\mathbb{Z} \subseteq B$ which is open. Then accordingly to (ii) of the previous corollary, there exists a ring of definition (A_0, I) for A such that $A_0 \subseteq B$. This makes B a f-adic ring.

Remarque 2 *Let A a topological ring.*
** A° is not necesseraly a subring. For instance, $A = \mathbb{R}, ||_\infty$ then $A^\circ = [-1, 1]$ is not a subring.*
**A° isn't necesseraly open, take $\mathbb{R}$ again, where $[-1, 1]$ isn't open. From what we've seen, these rwo properties are true for f-adic ring.*
**A° isn't bounded.*

exemple 1 *Take C non reduced, and then a non zero s t $a^n = 0$. Put $B = C[X]$, and $A = B_X = C[X, X^{-1}]$ with the induced structure of a Tate ring (cf exemple 1.1 (iv) of [5]) then $\frac{a}{X^m} \in A^\circ$ for all m, because $\frac{a}{X^m})^n = 0$, but if there existed a p s t $A^\circ.(X^p) \subseteq B = C[X]$, we would have $\frac{a}{X^{p+1}} X^p = \frac{a}{X} \in B$ which is absurd.*
But here A is not reduced.

exemple 2 *Put $B = k[X_i, X]_{i \geqslant 0}/(X_i^2 = X)$, and $A = B_X$ with the induced Tate structure. Put $a_n = \frac{X_1 \ldots X_{2n}}{X^n}$. Then $a_n^2 = 1$, so a_n^m depends only on the parity of m , and $a_n \in A^\circ$. So for every m , we have $a_{m+1} X^m = \frac{X_1 \ldots X_{2m+2}}{X} \notin B$.*
But here B isn't noetherian, nor integral $(X_1 - X_2)(X_1 + X_2) = 0$.

exemple 3 *case with B noetherian and integral ?*

Proposition 1.4 *Let A be a height 1 valuation ring. Then $k = qf(A)$, with the topology induced by A is a Tate ring.*

D : $A = k^\circ$ is an open subring, and A is adic with a finitely generated ideal of definition. Indeed, let $x \in k^{\circ\circ}$ and $x \neq 0$, i.e. such that $x \in \mathfrak{M}_A$, i.e. $0 < v(x) < 1$. Put $I = (x) = A.x$. Then I is an ideal of definition of A. Moreover, x is a nilpotent unit of k.

Définition 1.2 *A ring homomorphism $f : A \to B$ beetwen f-adic rings is called adic if there exist (A_0, I) and (B_0, J) rings of definition such that $f(A_0) \subseteq B_0$ and $f(I).B_0 = J$.*

Lemme 1.2 (1.8(i)) *If $f : A \to B$ is an adic ring homomorphism and $T \subseteq A$ is bounded, so is $f(T)$.*

D : let m , and so $J^m = B_0(f(I)^m)$ a 0 neighborhood. Then $\exists p$ such that $T.I^p \subseteq I^m \Rightarrow f(T)J^p \subseteq J^m$. □

Remarque 3 *If $f : A \to B$ is a ring homomorphism, then $f(A^{\circ\circ}) \subseteq f(B^{\circ\circ})$, and if f is adic, then from the lemma, $f(A^\circ) \subseteq B^\circ$: because if $\{a^n\}$ is bounded, so is $f\{a^n\} = \{f(a)^n\}$.*

1.1 Microbial valuation

Proposition 1.5 *Let (K, v) be a valued field. Then the topologies of $(K, +)$ having $U_g = \{x \in K \mid v(x) < g\}$, and $V_g = \{x \in K \mid v(x) \leqslant g\}$ as fundamental system of neighborhood of 0 make K a topological field and are the same.*

Définition 1.3 *Call the height of Γ the number of convex (called* isolated *in [2]) subgroups of Γ (possibly ∞).*
If A is a valuation ring, call the height of A, the height of its value group.
[2, prop 5 §4] , the height of A is the number of non-zero prime ideals of A, i.e. its Krull dimension

Proposition 1.6 (prop 8 §4 [2]) *Γ is of height 1 iff Γ is a subgroup of $(R, +, \leqslant)$.*

Définition 1.4 *[6, p. 39] a non archimedian field is a toplogical field whose topology is defined by a rank 1 valuation.*

Proposition 1.7 *Let K be a field, ν , ν' 2 valuations that are not unproper (unproper = trivial). According to [2, prop 3 §7] they define the same topology on K iff they are dependant, i.e. the ring generated by A_ν and $A_{\nu'}$ is not K.*

Let A be microbial, v the valuation it induces on A and $K = qf(A)$, then there exists , w another valuation, which is of height one such that they define the same topology. Let B the subring of K generated by $A = A_v$ and A_w. Then we have seen that $B \neq K$ and so [2, prop 1 §4] B is a valuation ring of K, let's call u its valuation. Then $A_w \subseteq B \neq K$, and [2, prop 4 §4] the subrings containing A_w correspond bijectively with the convex subgroups of $\Gamma_w \subseteq (\mathbb{R}, +)$. In that cas the only convex subgroups of Γ_w are $\{0\}$ and Γ itself, corresponding to the subrings A_w and K. So $B = A_w$ and $A_w \subseteq A$, i.e. we have proved that if A and B are dependant valuation ring and B is of hight 1, then $A \subseteq B$.
If A is a valuation ring, Γ its value group , $K = qf(A)$ there are correspondances :

$\{\mathfrak{p}$ prime ideals of $A\}$	$\leftrightarrow_\searrow$	$\{B \mid A \subset B \subseteq K, B$ subring$\}$	$\leftrightarrow_\nearrow$	$\{H \subseteq \Gamma$, convex subgroup$\}$
$\mathfrak{p}$	$\mapsto$	$A_\mathfrak{p}$		
$\mathfrak{m}_B \cap A = \mathfrak{m}_B$	$\leftarrow\!\shortmid$	B		
		B, $\exists \lambda : \Gamma_A \to \Gamma_B$ $s.t. v_B = \lambda \circ v_A$	$\mapsto$	$H_B = \mathrm{Ker}(\lambda)$
		$v_H = \lambda_H \circ v_A$ where $\lambda_H : \Gamma \to \Gamma/H$	$\leftarrow\!\shortmid$	H

These correspondances are [2, 3,§3 and 1 §4].
Hence A is of height 1 iff A is maximal for the subrings of K such that $A \subset B \subset K$ iff Γ_A doesn't have any convex subgroups except $\{0\}$ and Γ_A iff A is of Krull dimensio 1 , i.e. its only prime ideals are $\{0\}$ and $\mathfrak{m}_A$.

Proposition 1.8 *Let Γ be an orderd group. Then it has a convex subgroup $G \neq \Gamma$ maximal iff $\exists x \in \Gamma$ such that $< x >_{conv} = \Gamma$.*

D : $\Rightarrow$ Let $G \subsetneq \Gamma$ with G convex and maximal. Let $x \in \Gamma \backslash G$. The convex subgroups being totally orederd , and since $x \notin G$, $G \subsetneq < x >_{conv}$ so $< x >_{conv} = \Gamma$ because of the maximality of G.
$\Leftarrow$ Let $G = \cup_{H \subsetneq \Gamma \text{ convex}}$. Since convex subgroups are stable by union (for instance because they are totally orderd), G is convex. Since $x \notin H$ $\forall H$ in the union, $x \notin G$ hence $G \subsetneq \Gamma$, and is maximal for this property. ▫

Hence a valuation ring A is microbial
$\Leftrightarrow \exists A \subseteq B \subsetneq K$ with B of height 1
$\Leftrightarrow \exists A \subseteq B \subsetneq K$ with B maximal
$\Leftrightarrow A$ contains a prime ideal $\mathfrak{p} \neq 0$ minimal
$\Leftrightarrow \Gamma$ contains a convex subgroup maximal $\neq \Gamma$.
$\Leftrightarrow \exists g \in \Gamma$ such that $\Gamma = < x >_{conv}$.

Définition 1.5 *[6, p. 40] A valuation ring A is microbial if it satisfies one of the following equivalent property :*

1. *$qf(A)$ (with the toplogy induced by A) is a non archimedian field.*
2. *$qf(A)$ is a Tate ring.*
3. *$qf(A)$ has a topologically nilpotnet unit.*
4. *A is non-discrete and adic*
5. *A has a prime ideal of height 1.*

D : $1 \Rightarrow 2$ is proposition 1.4.
$2 \Rightarrow 3$ is in the definition of being a Tate ring.
$3 \Rightarrow 1$: let x be a nilpotent unit. Then $x^n \to 0$, and it is esay to see that $< x >_{conv} = \Gamma$ and we are done with the preceding remark.
$1 \Rightarrow 4$: Since A is of height 1 , it is not discrete ($\{0\}$ is not open) , and if B is a valuation ring of height 1 of $qf(A)$ that induces the same toplogy that A we can pick $x \in \mathfrak{m}_B$ small enough such that $x \in A$ (since A is a neigborhhod of 0) , and then we see that if $I = A.x$, then A is I-adic.
$4 \Rightarrow 1$: if A isn't discrete and adic. Let $i \in I \backslash \{0\}$ (this is possible precisely because A is not discrete so $I \neq \{0\}$. Then $\forall g \in \Gamma$, there exists a n such that $I^n \subseteq \{a \in A \mid v(a) < g\}$ hence $v(i)^n < g$ and using the fact that $v(A) \leqslant 1$ we have that $< i >_{conv} = \Gamma$.
$1 \Leftrightarrow 5$ was in the previous remark. □
example : Let $K = k(x, y)$, and $v_1 : K \to \mathbb{Z}^2_{lex}$
$P = \sum a_{(n,m)} x^n y^m \mapsto -\min((n, m) \mid a_{(n,m)} \neq 0\}$.
It is a valuation ([2, §3 , ex. 6] , with the general cas $v : k[\Gamma^+] \to \Gamma$, $\sum a_g x^g \mapsto -\min(g \mid a_g \neq 0$).
Let

$$\begin{array}{cccc} v_2 : & k(x,y) & \to & \mathbb{Z} \\ & \sum a_{n,m} x^n y^m & \mapsto & -\min\{n \mid \exists m \text{such that } a_{(n,m)} \neq 0\} \end{array}$$

Let $\pi : \mathbb{Z}^2 \to \mathbb{Z}$, $(n, m) \mapsto n$. Then $v_2 = \pi \circ v_1$.Let's call $\mathcal{T}_i$ the toplogies generated by v_i.
$\| \mathcal{T}_1 = \mathcal{T}_2$.
Let $V_{(n,m)} = \{f \in K \mid v_1(f) < (n, m)\}$ and $U_p = \{f \mid v_2(f) < p\}$.
Then $\forall (n, m)$, $U_{n-1} \subseteq V_{(n,m)}$ so $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Conversly , $\forall (n, m), V_{(n,m)} \subseteq U_n$ so $\mathcal{T}_2 \subseteq \mathcal{T}_1$.□
In fact since $v_2 = \pi \circ v_1$, $A_{v_1} \subseteq A_{v_2}$ and they are proper valuation, so cf prop 1.7 , they define the same toplogy. In this exemple , A_{v_2} is not a valuation of height 1, but it is microbial.

a valuation not micorbial
Let $\Gamma = \mathbb{Z}^{(\mathbb{N})}$ (the sequences in $\mathbb{Z}$ with finite support), with the (reverse) lexicographic order, i.e. $x = (x_0, \ldots x_n, 0 \ldots)$ with $supp(x) \subseteq \{0 \ldots n\}$, and the same thing for y , if $x_n > y_n$ then $x > y$. More generally , if $x_i = y_i$ for $i > n$ and $x_n > y_n$ $x > y$. We can then define
$v : k(x_i)_{i \in \mathbb{N}} \to \mathbb{Z}^{(\mathbb{N})}$ by $v(\sum a_\nu x^\nu) = -\min(\nu \mid a_\nu \neq 0)$. It is easy to see that the convex subgroups of $\Gamma = \mathbb{Z}^{(\mathbb{N})}$ are the $\Gamma_n = \{x \mid supp(x) \subseteq \{0 \ldots n\}\} = < x >_{conv}$ for any x of the form $x = (x_0, \ldots, x_n, 0, 0 \ldots)$ with $x_n \neq 0$, and hence there doesn't exist a proper maximal convex subgroup. Hence A_v is not micorbial. (we could also have seen, that for any x , $< x >_{conv} \neq \Gamma$)

Remarque 4 *We can extend the definition of being microbial to fields (this is actually nothing since a field is a valuation ring) , and to valued ring $v : A \to Gamma$, by saying that if $B = A/supp(v)$ and $K = qf(B)$, K is microbial. All the preceding properties work as well.*

Here is an exemple with

$$\begin{array}{ccc} A & \xrightarrow{v} & \Gamma_0 \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle i} \\ B & \xrightarrow{w} & H_0 \end{array}$$

with $v \sim w \circ \phi$, w microbial but v not microbial.
Inded take v not microbial on $A = k$ a field. Then put $B = k[x]$, $H = \mathbb{Z} \times \Gamma$ and $w(\sum a_i X^i) = \max((-i, v(a_i)) \mid a_i \neq 0)$. It is a microbial valuation on B for instance $< (1, 0) >_{conv} = H$, or $0 \times \Gamma$ is a maximal proper convex subgroup).

2 Valuation Spectrum

2.1 compacity, filters

cf[3, §6,7,9]
Let X be a topological space, $\mathcal{F}$ a filter on X , $x \in X$, $\mathcal{B}(x)$ the filter of neigborhhod of x. We say that $\mathcal{G}$ is finer than $\mathcal{F}$ if $\mathcal{G} \supseteq \mathcal{F}$

Définition 2.1 *x is a limit point of $\mathcal{F}$ if it is finer than $\mathcal{B}(x)$, i.e. every neigborhhod contains an element of $\mathcal{F}$.*

$\mathcal{B}$ is said to be a base of filter if it is stable by finite intersection, and doesn't contains $\emptyset$.

Définition 2.2 *Let $\mathcal{B}$ be a base of fliter, x is adherent to $\mathcal{B}$ if for every $B \in \mathcal{B}$, $x \in \bar{B}$.*

If $\mathcal{F}$ is finer than $\mathcal{G}$ and x adherent to $\mathcal{F}$, then it is also adherent to $\mathcal{G}$.

Proposition 2.1 (§6 , cor 2) *Let $\Phi = \{\mathcal{F}_\rangle\}$ a set of filter. There exists a filter finer that all the $\mathcal{F}_i$ iff for all $\mathcal{F}_1 \dots \mathcal{F}_n \in \Phi$ and $F_i \in \mathcal{F}_i$, $F_1 \cap \dots \cap F_n \neq \emptyset$.*

So x is adherent to $\mathcal{F}$
$\Leftrightarrow \forall F \in \mathcal{F}$ and $U \in \mathcal{B}(x)$, $F \cap U \neq \emptyset$.
$\Leftrightarrow \exists$ a filter $\mathcal{G}$ finer than $\mathcal{F}$ and $\mathcal{B}(x)$. Indeed consider for $U \in \mathcal{B}(x)$ the filter $\mathcal{F}_U = \{V \| U \subseteq V\}$, and apply the proposition with $\Phi = \{\mathcal{F}_U \mid U \in \mathcal{B}(X)\} \cup \{\mathcal{F}\}$.
$\Leftrightarrow \exists$ a filter $\mathcal{G}$ finer thant $\mathcal{F}$ which converges to x.

Corollaire 2.1 *Let $\mathcal{U}$ be a ultrafilter. $\mathcal{U}$ converges to x iff x is adherent to it.*

Définition 2.3 (Prop) *X is quasi compact if it satisfies one of the following properties :*

1. *every filter has an adherent point*
2. *every ultrafilter is convergent*
3. *Every family of closed set whose intersection is empty has a finite subfamily whose intersection is empty.*
4. *Every open cover has a finite subcover.*

D : (i) $\Rightarrow$ (ii) Let $\mathcal{U}$ be a ultrafilter, it has an adherent point, and so converges to it.
(ii) $\Rightarrow$ (i) : let $\mathcal{F}$ be a filter, $\mathcal{U}$ a ultrafilter which is finer, it converges to x say , so x is adhrent to $\mathcal{U}$ and also to $\mathcal{F}$.
(i) $\Rightarrow$ (iii) Let $\{F_i\}$ be a family of closed subsets whose intersection is empty. Let's suppose that every finite intersection is non empty. Then there exists a filter $\mathcal{F}$ that contains all the F_i. Let x be an adherent point, so $x \in \bar{F_i} = F_i$ for all i, which contradicts $\cap_i F_i = \emptyset$.
(iii)$\Rightarrow$ (i) Let $\mathcal{F}$ be a filter , and suppose it has no adherent point. Then $\forall x \in X, \exists F_x \in \mathcal{F}$ with $x \notin \bar{F_x}$, and since $F_x \in \mathcal{F}$, $\bar{F_x}$ too. So the $\bar{F_x}$ have the finite intersection property , however by construction, their intersection is empty, which contradicts (iii).
(iii) and (iv) are dual. □

2.2 remark on compacity

A (open) basis of X is a family $\mathcal{B}$ (of open subsets necesseraly from the following definition) , such that the open of X are the (arbitrary) union of elements of $\mathcal{B}$. Dually, it will be called a closed basis, if the closed sets are the intersection of elements of $\mathcal{B}$.
A (open) sub-basis of X is a family $\mathcal{C}$ (of open subsets necesseraly from the following definition) , such that the open of X are the (arbitrary) union of finite intersection of elements of $\mathcal{C}$. Dually, it will be called a closed sub-basis, if the closed sets are the intersection of finite union of elements of $\mathcal{C}$. The family $\mathcal{B}$ of finite intersection of $\mathcal{C}$ is then clearly a basis, called the basis generated by $\mathcal{C}$.
Let $\mathcal{C}$ be a subbasis, and $\mathcal{B}$ the basis it generates. Taking the complementary, we give the same name to the (sub)-basis of closed or open sets by taking the complementary
Then the following are equivalent :
X is quasi compact
$\Leftrightarrow$ Every open cover has a finite subcover
$\Leftrightarrow$ Every open cover by elements of $\mathcal{B}$ has a finite subcover.
$\Leftrightarrow$ Every family of closed set of $\mathcal{B}$ whose intersection is empty has a finite subfamily whose intersection is empty.

Proposition 2.2 *The following are equivalent :*

1. *Every family of closed set of $\mathcal{B}$ whose intersection is empty has a finite subfamily whose intersection is empty.*
2. *Every family of closed set of $\mathcal{C}$ whose intersection is empty has a finite subfamily whose intersection is empty.*

D : clearly since $\mathcal{C} \subseteq \mathcal{B}$ (i) $\Rightarrow$ (ii).
Let's suppose (ii), and let $\mathcal{F} = \{F_i\}$ be a family of closed subsets of $\mathcal{B}$ with the finite intersection property. Let's show that $\cap_i F_i \neq \emptyset$.
Let A be a maximal family of closed subsets of $\mathcal{B}$ such that $A \supseteq \{F_i\}$ and has the finite intersection property. (such an A exists with Zorn's Lemma). So $\cap_{F \in A} F \subseteq \cap_i F_i$ so it is enough to show that $\cap_{F \in A} F \neq \emptyset$. We now suppose $\{F_i\}$ maximal. It is eqsy to see that it implies that the family is stable by finite intersection. Every F_i can be written : $F_i = F_i^1 \cup \ldots F_i^n$ (should write n_i instead of n...) Let's show that for every i there exists a j with $F_i^j \in \mathcal{F}$
Let $j \in \{1 \ldots n\}$. If $\forall G \in \mathcal{F}\ G \cap F_i^j \neq \emptyset$ then the family $\mathcal{F} \cup \{F_i^j\}$ still has the finite intersection property and we are done. Otherwise, for all j there exists a $G_j \in \mathcal{F}$ such that $G_j \cap F_i^j = \emptyset$. Then $G = \cap_j G_j \in \mathcal{F}$, but $\forall j$, $G \cap F_i^j = \emptyset$ so $G \cap F = \emptyset$, which is a contradiction (with the FIP).
So $\forall i$, there exists a j_i such that $F_i^{j_i} \in \mathcal{F}$. Then $\cap_i F_i^{j_i} \subseteq \cap_i F_i$ and since by construction $f_i^{j_i} \in \mathcal{C}$ (ii) implies that $\cap_i F_i^{j_i} \neq \emptyset$, so $\cap_i F_i \neq \emptyset$.

Here is another proof : let's show that X is quasi compact, i.e. satisfies the property, every ultrafilter converges to some x. Indeeed le t$\mathcal{U}$ be a ultrafilter and let's suppose it doesn't converge to any x. Then for every x we can find $F \subseteq \mathcal{U}$ such that $x \notin \bar{F}$. Then there exists $F_1 \ldots F_n$ some closed of $\mathcal{C}$ such that $G = F_1 \cup \ldots \cup F_n \supseteq \bar{F}$, and $x \notin F_1 \cup \ldots F_n = G$. Then $G \in \mathcal{U}$ so there exists one i such that $F_i = F_x \in \mathcal{U}$ since it is an ultrafilter. But the $F_x \in \mathcal{C}$, they have the FIP, but have empty intersection since $\forall x, x \notin F_x$.
□
We can then deduce that :

Proposition 2.3 *X is quasi compact*
$\Leftrightarrow$ Every family of closed set of $\mathcal{C}$ whose intersection is empty has a finite subfamily whose intersection is empty.
$\Leftrightarrow$ Every open cover by elements of $\mathcal{C}$ has a finite subcover.

2.3 constructible sets

cf EGA_0 §9.

Définition 2.4 *$Z \subseteq X$ is retrocompcat iff $\forall U$ qc open, $Z \cap U$ is qc (note that it is equivalent that $Z \cap U$ is qc in Z or in U since this only depend on the topology of $Z \cap U$), i.e. if $i : Z \subseteq X \hookrightarrow X$ is quasi-compact.*

Définition 2.5 *$S \subseteq X$ is constructible if it is in the boolean algebra $(\cap, \cup, ^c)$ generated by the open retrocompcat.*

Proposition 2.4 *Let $V \subseteq X$ retrocompcat and U open in X , then $V \cap U$ is retrocompact in U.*

D : let $W \subseteq U$ a qc open. Then $(V \cap U) \cap W = V \cap W$. Since W is qc in X, and V retrocompact, $V \cap W$ is qc in X , so also in U. □

cf rq après 9.1.1 :

Remarque 5 *1. if V_1 and V_2 are retrocompact, $V_1 \cup V_2$ too.*

2. *If V_1, V_2 are retrocompact open, then $V_1 \cap V_2$ too.*

Indeed; (i), if $U \subseteq X$ is open qc, $V_1 \cap U$, and $V_2 \cap U$ are qc. Quasi compact sets are stable by finite union so, $(V_1 \cup V_2) \cap U$ is qc.
(ii) If $U \subseteq X$ is qc open , $V_1 \cap U$ is qc open, so $V_2 \cap (V_1 \cap U)$ too.

This is probably the reason why in EGA , the retrocompact sets are introduced, because, the retrocompact open are stable by intersection, wheras qc not necesseraly (unless you make the assumption X is quasi-separated...which is tautological).

If X is Haussdorf, U is qc open, iff U is compact open iff U is compact open-closed.

Proposition 2.5 (EGA0 9.1.8) *If $U \subseteq X$ is open.*

1. *If T is constructible in X, $T \cap U$ is constructible in U*
2. *If U is in addition retrocompact, the converse is true : if $T \subseteq U$ is constructible in U, it is also constructible in X.*

Définition 2.6 *$T \subseteq X$ is locally constructible, if for every $x \in X$ there exists V an open neighborhood of x such that $T \cap V$ is constructible in X.*

Définition 2.7 (EGA$_4$ 1.9) *$E \subseteq X$ is pro-constructible (resp. ind-constructible) if for every $x \in X$ there exsists V a neigborhood of x such that $V \cap E$ is an intersection of locally constructible sets (resp union).*

Remarque 6 (cf EGA0 9.1.11) *If $U \subseteq X$ is open, and T locally constructible in X, then $U \cap T$ is locally constructible in U.*
(EGA0 9.1.10) If X is quasi compact, and has a basis of open retrocompact, then T is constructible iff it is locally constructible.
(EGA4 1.9.4) Under these hypothesis, $T \subseteq X$ is pro-constructible iff it is an intersection of constructible : indeed then we can cover X by some finite retrocompact open (since retrocompact open $\Rightarrow$ qc) , X_i, say $T = \cup_i T \cap X_i$, and $T \cap X_i = \cup_j T^i_j$ is constructible in X , then $T = \cup_i \cap_{j \in J_i} T^i_j = \cap_{(j_1,\dots,j_n) \in J_1 \times \dots J_n} \cup_{i=1\dots n} T^i_{j_i}$ is an intersection of constructible sets.

2.4 spectral spaces

Définition 2.8 *X is quasi-separated if for every qc open U and V , $U \cap V$. is qc*
Said differnetly, X is quasi-separated iff the qc open are retrocompact.

Moreover, if X is quasi-compact quasi-separated, the qc open are precisely the retrocompact open. To give a counter-exemple, let $X = \mathrm{Spec}(k[T_i]_{i \geqslant 0})$, and $U = X \backslash \{(T_i)_i\}$. Define Y as two copies of X (say X_1 and X_2 , glued along U. Then like X, X_i are qc, but $X_1 \cap X_2 = U$ isnot qc.

Remarque 7 *Let X be a topological space such that there exists a basis for the topology $\mathcal{B} = \{U\}$ which are qc, and stable by finite intersection. Then, if V, W are qc open of X , $V \cap W$ is also qc. Indeed write $V = \cup_{i=1..n} V_i$ with $V_i \in \mathcal{B}$. (this is possible because V is qc and $\mathcal{B}$ a basis. Do the same for W, then $V \cap W = \cup_{i,j} V_i \cap W_j$ is then a finite union of qc sets, so is qc. Hence X is quasi-spearated.*
If X is a separated scheme, it verifies these hypothesises, so the intersection of two quasi compact is quasicompact.

Définition 2.9 *[4, 0] X is spectral if it is T_0 , quasi-compact , the qc open form a basis and are stable by finite intersection, and every non empty closed irreducible subset has a generic point.*

Remarque 8 *In [5, 2] the definition is with a unique generic point, but without T_0. This is equivalent : suppose that X is T_0 if $\bar{x} = \bar{y}$ $x \neq y$, let U be an open s t $x \in U, y \notin U$. Then $x \in \bar{y} \subseteq U^c$ contradiction. Conversly if the generic points are unique, let $x \neq y$, then $\bar{x} \neq \bar{y}$, say, $\bar{x} \nsubseteq \bar{y}$, it implies $x \notin \bar{y}$, then $x \in \bar{y}^c$ which separates x and y.*
From the previous remark, if X is spectral, X is quasi-separated.
It also implies, that in the definition, you can only require that there exists a basis of the topology which with qc open, which are stable by finite intersection (this is the statement ([4] , prop4 (i) $\Leftrightarrow$ (ii)).

Proposition 2.6 *Let X be a spectral space. An open U is retrocompact iff it is quasi compact.*

D : $\Rightarrow$ If U is retrocompact, since X is qc, $X \cap U = U$ is qc.
$\Leftarrow$: Let V be an open qc. Then $U \cap V$ is qc. ▫

In the following part, X will always be a spectral space.

Proposition 2.7 *$T \subseteq X$ is locally constructible iff T is constructible.*

D : T loc constructible iff $\forall x\ \exists V_x$ $x-$neigborhhod, such that $T \cap V_x$ is constructible in V_x
$\Leftrightarrow \exists X = X_1 \ldots \cup X_n$ such that $T \cap X_i$ is constructible in X_i, and X_i qc , using the fact that qc open form a basis, and that X is qc, and that interecting with an open preserves constructible sets
$\Leftrightarrow$ T constructible, since the X_i being quasi-compact, they are retrocompact, and then $T \cap X_i$ constructible in X_i implies it is constructible in X , and $T = \cup_i(T \cap X_i)$. ▫

Proposition 2.8 *T is proconstructible iff T is an intersection of constructible sets in X.*

We only have to show $\Rightarrow$.
T proconstructible iff $\forall x\ \exists V_x$ such that $E \cap V_x$ is an intersection of locally constructible in V_x
$\Leftrightarrow \forall x \exists V_x$qc such that $T \cap V_x$ is an intersection of locally constructible. (using the fact that qc form a basis, and the fact that locally constructible are preserved by intersecting with an open, so intersection of locally constructible are preserved when intersecting with an open)
$\Leftrightarrow \forall x\ \exists V_x$ open qc such that $T \cap V_x$ is an intersection of constructible (using the fact that V_x is an open retrocompact of X
$\Leftrightarrow X = X_1 \cup \ldots X_n$ with X_i qc and $T \cap X_i$ intersection of constructible in X_i. Then since X_i is qc so retrocompact , we see that $T \cap X_i$ is an intersection of constructible of X, say $T \cap X_i = \cap C_{i,j}$.
Then $T = \cup_{i=1..n}(\cap_{J_i} C_{i,j}) = \cap_{J_1 \times \ldots J_n} \cup_{i=1..n} C_{i,j}$ which is an intersection of constructible of X.

Remarque 9 *So what [4] calls the patch topology X_{patch}, is what EGA4 1.9.13 , calls the constructible topplogyX_{cons}. The open subsets are the ind-constructible subsets , and the closed pro-constructible.*

Proposition 2.9 *X_{cons} is compact.*

D : It is Haussdorf, because $\exists$ open qc U that separates two points x, y , so U and U^c are open that separate x, y.
If whe use the remark on compacity, let's note $\mathcal{C}$ the subbasis of closed sets of X_{cons} formed by (arbitrary) closed and qc open (from X). Then we have to check that a family A of $\mathcal{C}$ which has FIP has non empty intersection. WIth Zorn,if we take B a maximal family with the FIP containing B , its intersection will be smaller than that of A , so we can restrict to B , i.e. suppose that A is maximal with the FIP.
$A = \mathcal{F} \cup \mathcal{U}$, the closed, and the qc open. Then, because X is quasi-compact , $G = \cap_{F \in \mathcal{F}} F$ is a closed non empty. Then G has the FIP $\mathcal{F}, \mathcal{U}$ (for the qc open , this is because the $F_i \cap U$ have the FIP , that U is qc so their intersection, which is $G \cap U$ is non empty), so by maximality , $G \in A$. If it wasn't irreducible , let's write it $G = G_1 \cup G_2$ .Then if $A \cup \{G_i\}$ $i = 1, 2$ doesn't have the FIP , we would have an $A_i \cap G_i = \varnothing$, whence $G \cap (A_1 \cap A_2) = \varnothing$ but $A_1 \cap A_2 \in A$ absurd. So say $G_1 \in A$ so $G_1 = G$ and G is irreducible, say $G = \bar{g}$, then $g \in F$ for all closed. and if $g \notin U$ for one open , $\bar{g} \cap U = \varnothing$, absurd.
second proof : Let $\mathcal{F}$ be a family of $\mathcal{B}$, the closed-basis of X_{cons} of $pro-constructible$ sets formed by the $F \cap U$, F closed, and U qc open, with the FIP. With Zorn's Lemma, we can assume it is maximal. Then for $F \cup U \in \mathcal{F}$, F or $U \in \mathcal{F}$, indeed oterwise, there are $A, B \in \mathcal{F}$ with $A \cap F = \varnothing$ and $B \cap U = \varnothing$, then $F \cap A \cap B = \varnothing$ which contradicts FIP. Let then $\mathcal{F}_1$ be the closed sets of $\mathcal{F}$ and Let then $\mathcal{F}_2$ be the open sets of $\mathcal{F}$. One has $\cap_{\mathcal{F}} A = \cap_{\mathcal{F}_1 \cup \mathcal{F}_\in} A$. Then $F = \cap_{\mathcal{F}_\infty} A$ is a non empty closed set (by hypothesis). It is irreducible , bacause if $F = F_1 \cup F_2$ with the same argument that above one shows one of the F_i is F. So since X is spectral, $F = \overline{\{x\}}$ for some $x \in X$, and as above one shows $x \in \cap_{\mathcal{F}} F$. ▫

Corollaire 2.2 *If X is spectral, the constructible subsets of X are exactly the closed-open subsets of X_{cons}*

D : $\Rightarrow$ If U is qc open in X, by definition, it becomes a closed open of X_{cons}, and since the closed-open are stable by finite boolean combination we are done.
$\Leftarrow$ Let U be a closed open of X_{cons}. It is then compact , since closed in a compact. Since by definition the $U \cap V^c$ form a basis of X_{cons} for U, V qc open of X, we can write $U = \cup_{i=1...n} U_i \cap V_i^c$ with U_i, V_i qc open, so U is constructible. □

Proposition 2.10 ([4] prop 4) *Let X be quasi-compact, T_0, has a basis formed by qc open that are closed under finite intersecion. The following are equivalent :*

1. *X is spectral*
2. *Every nonemty irreducible closed subspace has a generic point*
3. *every family of qc open of a closed subspace with the FIP has finite intersection.*
4. *X_{cons} is compact and has a basis of closed-open sets.*
5. *X_{cons} is quasi-compact*
6. *A family of pro-constructible sets with the FIP has non empty intersection.*

D : (i) $\Leftrightarrow$ (ii) this is a consequence of 7.

(i) $\Rightarrow$ (v) is 2.9
(v) $\Rightarrow$ (vi) is just the alternative definition of quasi-compacity , and the fact that the pro-constructible are the closed sets of X_{cons}.
(vi) $\Rightarrow$ (iv) X_{cons} is then quasi-compact. Since the qc open form a basis of X, X_{cons} is Haussdorf (so compact), and in fact by definition, the sets of the form $F \cap U$ with U qc open, and F the complementary of a qc open form by definition a basis of X_{cons}. Their complementary is $F^c \cup U^c$ are also open in X_{cons} , so $F \cap U$ is close-open.
(iv) $\Rightarrow$ (iii) : Let F be a closed set and $\{U_i\}$ a family of quasi-compact open of F with the FIP. Each U_i is qc , $U_i = F \cap V_i$ where V_i is open in X. Hence since the qc open form a basis of X, we can write $V_i = \cup_{J_i} W_j$, and since each U_i is quasi-compact, there esists a finite subset (say $\{1 \ldots n\}$ such that $U_i = \cup_{1...n} W_j \cap F$. Hence, each U_i is proconstructible in X , and since X_{cons} is compact 2.9, they have non empty intersecion.
(iii) $\Rightarrow$ (ii) Let F be an irreducible closed subset. Put $G = \cap_{U \text{non-empty qc open of} F} U$. It is non empty (a space Z is irreducible $\Leftrightarrow$ finite intersecions of non empty open are non empty) , so the set of non-empty qc open of F has FIP and we use the hypothesis.
Suppose $x \neq y \in G$. Then, $\exists U$ a qc open of X such that say $x \in U$ and $y \notin U$ (because X is T_0 and there is a basis of qc open. Then $U \cap F$ is qc open and non empty, so $y \notin G$ which is absurd. So $G = \{x\}$. Suppose $\overline{\{x\}} \neq F$. Then $V = F \backslash \overline{\{x\}}$ is a non empty open of F so contains a non empty qc open of V of F, but $x \notin V$ contradiction. □.

Proposition 2.11 ([4] Prop 7 , cf also [5] (rem 2.1 (vi)) *Let $(X, \mathcal{S})$ a compact space , $\mathcal{B} = \{U\}$ a family of closed-open sets (hence compact) of X. Let $\mathcal{T}$ the topology of X which has $\mathcal{B}$ as a sub-basis.*
Then $(X, \mathcal{T})$ is T_0 $\Leftrightarrow$ $(X, \mathcal{T})$ is spectral, and in that case the constructible subsets of $(X, \mathcal{T})$ are precisely the closed-open subsets of $(X, \mathcal{S})$

D : $\Rightarrow$ is clear .
$\Leftrightarrow$ Taking finite intersection of $\mathcal{B}$ doesn't change the fact it is formed by closed-open sets of $(X, \mathcal{S})$, so we can assume $\mathcal{B}$ is a basis stable under intersection of $\mathcal{T}$.
By definition, $\mathcal{T} \subseteq \mathcal{S}$, hence it remains quasi-compact, and has a basis ($\mathcal{B}$) stable under intersection of quasi-compact open and is T_0 by hypothesis. So according to 2.10 we just have to prove that $(X, \mathcal{T})_{cons}$ is compact.
Now, let V be a quasi-compact open of $(X, \mathcal{T})$, so by a quasi-compacity $V = \cup_{i=1...n} U_i$ with $U_i \in \mathcal{B}$. And $V^c = \cap_{i=1...n} U_i^c$. We can deduce from that :
$\iota : (X, \mathcal{S}) \rightarrow (X, \mathcal{T})_{cons}$ is continuous. Since it is bijective, and $(X, \mathcal{S})$ is compact and $(X, \mathcal{T})_{cons}$ is Hausdorff, it is a homeomorphism (the direct image of a closed is the direct image of a compact so compact). So in fact $(X, \mathcal{S}) = (X, \mathcal{T})_{cons}$ which is then compact
The fact that constructible of $(X, \mathcal{T})$ are the closed-open of $(X, \mathcal{S})$ is then 2.2 . □

Remarque 10 *X a spectral space ; T proconstructible. Then [5, 2.1(i)] T is qc in the topology of X and X_{cons}. In particular, since X is constructible, it is quasi-compact in X_{cons}. Note that , X_{cons} is Haussdorf : indeed, X is T_0, so if $x \neq y$, let say U a neighborhood of x not containing y. Since X is spectral, we can assume U is qc , so U and U^c are open in X_{cons} and separate x and y. So , X_{cons} is compact.*

Définition 2.10 *(cf [4, 0] or [5, 2.2.1]) a map $f : X \to Y$ between spectral spaces is said spectral if it is continuous and f^{-1} preserves the qc open (which actually implies continuity)*

Proposition 2.12 *f is spectral iff f is continuous and $f : X_{cons} \to Y_{cons}$ is continuous.*

$\Rightarrow$ let V be an open of Y_{cons} , i.e. a union $\cup_i V_i$ with each V_i constructible, i.e. boolean combination of qc open. Since f^{-1} commutes with boolean combination and preserves qc open $f^{-1}(V_i)$ is constructible.
$\Leftrightarrow$: if V is a qc open, it is constructible and $f^{-1}(V)$ is constructible open, so ([5, 2.1 (i)]) open qc.
□

Proposition 2.13 (Dickmann p.90) *Let $f : X \to Z$ and $g : Y \to Z$ spectral maps. Then $\pi : X \times_Z Y \to Y$ (taken in Top) is spectral.*

D : π factrizes as : $X \times_Z Y \xrightarrow{a} X \times Y \xrightarrow{b} Y$. $X \times_Z Y$ is closed, so proconstructible in $X \times Y$, so [D 3.3.1] a is spectral. with [D p88] , b is spectral too. □
Are qc map in Top stable by base change ?

Définition 2.11 *A topological space X is locally spectral if there exists a covering X_i such that each X_i is spectral.*

Remarque 11 *In [4] Theorem 9, it is proved that locally spectral spaces are precisely the underlying topological spaces of schemes (what he called prescheme).*

Proposition 2.14 (cf [6] p44) *A locally spectral space X is spectral iff it is quasi-separated and quasi-compact.*

D : $\Leftarrow$ is obvious. Then suppose X locally spectral, quasi-compact and quasi-separated. Then cover it with $X_i, i = 1 \dots n$ that are spectral. For each i consider the inclusion : $f_i : X_i \to X$. If U is a qc open of X, then $f_i^{-1}(U) = U \cap X_i$ is quasi compact (since X is quasi-separated). We deduce that $f_i : X_{i cons} \to X_{cons}$ is continuous. Then since one easily sees that $(X_1 \coprod \dots X_n)_{cons} = X_{1cons} \coprod \dots \coprod X_{ncons}$, one sees from the second one that it is compact, and $f : (X_1 \coprod \dots X_n)_{cons} \to X$ is continuous and surjective, so X_{cons} is compact. Now, X is T_0 , quasi compact and quasi-separated, (this is local property (contrary to being T_2) , so [4, Prop 4 (v)] X is spectral.□

Proposition 2.15 *If $f : X \to Y$ is spectral , and $T \subseteq X$ proconstructible, then $f(T)$ is proconstructible.*

D : T proconstructible mean T closed in X_{cons}, since f spectral $\Leftrightarrow$ $f : X_{cons} \to Y_{cons}$ continuous and that X_{cons} and Y_{cons} are compact, $f(T)$ is compact, so closed in Y_{cons}, so proconstructible. □

Proposition 2.16 *if $f : X \to Y$ is spectral and surjective, $S \subseteq Y$ is constructible (resp. proconstructible) iff $f^{-1}(S)$ is.*

$\Rightarrow$ is OK .
Conversly , by surjectivity of f , we have $S = f(f^{-1}(S))$ and $S^c = f(f^{-1}(S)^c)$. If $f^{-1}(S)$ is constructible i.e. closed open in X_{cons}, then $f^{-1}(S)^c$ is closed to , their images are then S , S^c , which are both closed in Y_{cons} , so S is closed-open in Y_{cons}, so constructible in Y. And if $f^{-1}(S)$ is proconstructible, $S = f(f^{-1}(S)$ is proconstructible too. □

Proposition 2.17 *(Dickmann, scwartz Tressl, Spactral Spaces, theorem 3.3.1) . Let X be a spectral space, and $T \subseteq X$. Then T is proconstructible iff T, with the induced topoloy, is spectral, and $i : T \hookrightarrow X$ is a spectral map. Moreover constructible (resp qc open, resp complementary of qc open) in T , are the traces of constructible(resp. qc open, resp complementary of qc open) in X.*

From that we deduce somme consequences of [5, 2.1] when T is a proconstructible of X
(i) T is quasi-compact for X and X_cons. In particular an open subset is constructible iff it is qc, and a closed subset is constructible iff its complementary is quasi compact (as a closed subset of a quasi-compact set, it is anyway quasi-compact). Indeed $T = i^{-1}(X)$ with X qc and i spectral so T is qc in X. And T is closed in X_{cons} so compact in X_{cons}.
(ii) T is constructible iff T^c is proconstructible. Indeed $\Rightarrow$ is clear, and if T^c is proconstructible, T is closed-open in X_{cons} so constructible 2.2.
(iii) $\overline{T} = \cup_{t\in T}\overline{\{t\}}$.
Let $x \in \overline{T}$ and $\mathcal{U} = \{U \mid U$ is a quasi-compact open neigborhhod of $x\}$. Hence $x \in \cap_{U\in\mathcal{U}}U$. Since $x \in \overline{T}$, $\forall U \in \mathcal{U}$, $U \cap T \neq \emptyset$. More generally, if $U_1 \ldots U_n \in \mathcal{U}$, $U_1 \cap \ldots \cap U_n \cap T = (u_1 \cap Y) \cap \ldots \cap (U_n \cap T) \neq \emptyset$. So the $\{U \cap T\}_{U\in\mathcal{U}}$ have the FIP , and are proconstructible in X since T is and the U are, so their intersection is not empty (since X_{cons} is compact). Let then $t \in \cap_{U\in\mathcal{U}}(U \cap T)$, and $V = \overline{\{t\}}^c$ which is then open. If $x \in V$, there exists $U \in \mathcal{U}$ such that $x \in U \subseteq V$. But by hypothesis $t \in U \subseteq V$ which is absurd. So $x \in \overline{\{t\}}$.

2.5 Valuations

Proposition 2.18 ([2] , Prop 9 §3) *A valuation ring A is noetherian iff a discrete valuation ring (i.e. $\Gamma_v = \mathbb{Z}$) iff principal (because a finitely generated ideal of a valuation ring is principal)*

For instance take $v : k(x_1, x_2) = A \to \mathbb{Z}^2_{lex}$
$\sum a_\nu x^\nu \mapsto -\min(\nu \mid a_\nu \neq 0\}$. Then R the associated valuation ring is not noetherian. Indeed, its ideal correspond to the interval of $\mathbb{Z}^{2\ +}$. Among them $\cup_{n\geqslant 0}]-\infty,(-1,n)]$ is not principal, i.e. of the form $]-\infty, a]$. This ideal is $R.(\frac{x_1}{x_2^n})_{n\geqslant 0}$.

Remarque 12 *Let $v : A \to \Gamma_0$ a valuation, R its valuation ring of the residual field of v. What is the link beetwen A and R being noehterian ?*
If $R \subset K$ is a vualtion ring and $R \subseteq A \subseteq K$ an intermediate valuation ring, what link betwen A and R being noehterian ?
No link because R will be noetherian iff $\Gamma = \mathbb{Z}$. So for the first question, take $v : A = \mathbb{C}_p[T] \to (\mathbb{Q}, +, \leqslant)$, $\sum a_i T^i \mapsto \max(|a_i|)$ A is noehterian but $\Gamma \neq \mathbb{Z}$, i.e. A noetherian but not R.
On the contrary, $v : A = k[x_i]_{i\in\mathbb{N}} \to \mathbb{Z}$, $\sum A_i x_0^i \mapsto -(\min(i \| A_i \neq 0)$ where the $A_i \in k[x_1, \ldots]$.
Now if $R \subseteq A \subseteq K$ then if R is noehterian it is a discrete valuation ring , and the only possiblities for A are R and K.
But if R is not noetherian, there will exist a intermediate A noetherian (ie . discrete valuation ring) iff there exists a quotient $\Gamma/H \simeq \mathbb{Z}$ with H necesseraly a (the) maximal proper conves subgroup. Sometimes it is the case, for instance $v : k[x, y] \to \mathbb{Z}^2, \sum a_{(n,m)} x^n y^m \mapsto -\min((n, m) \mid a_{(n,m)} \neq 0\}$, sometimes not, for instance in the case of a non microbial valuation.

Proposition 2.19 *Let $v : A \to \Gamma_0$ a valuation with $\Gamma = \Gamma_v$, and $\alpha : \Gamma \to G$ such that $v \sim \alpha \circ v$. Then v is injective.*

D : Otherwise let $h \in \mathrm{Ker}(\alpha)\backslash\{1\}$. Then there exists $x \in K$ the residual field of v with $v(x) = h \neq 1$, but the same calculus in K_w leads $v(x) = 1$. ▫

Remarque 13 *A valuation on A : $v : A \to \Gamma_0$ is equivalent to give $\mathfrak{p} = v^{-1}(0)$ a prime ideal of A and a valuation v on $qf(A/\mathfrak{p}$ which is also equivalent to give an equivalent class of morphism $A \xrightarrow{\phi} k$ with k a valued field, where $(\phi, k) \sim (k', \phi')$ if there exists an morphism of valued field $\iota : k \to k'$ such that $\phi' = \iota \circ \phi$.*

2.6 Embedding in $\mathcal{P}(A \times A)$

Proposition 2.20 (2.2) *Let A be a ring, $|$ a binary relation on A such that*

1. *$a|b$ or $b|a$ $\forall a, b$.*
2. *If $a|b$ and $b|c$ then $a|c$.*
3. *$a|b$ and $a|c$ implies $a|b+c$.*
4. *$a|b$ implies that $ac|bc$*
5. *$ac|bc$ and if not $0 \nmid c$ then $a|b$.*
6. *$0 \nmid 1$.*

Then there exists a unique equivalence class of valuation v s t $| = |_v$ where $a|_v b$ iff $v(a) \geqslant v(b)$.

D : Let $\sim$ be the e binary relation defined by $a \sim b$ iff $a|b$ and $b|a$.
This is an equivalence relation . Indeed reflexivity is a consequence of (i) , transitivity from (ii) , and symetry is obvious.
Let's not $\mathfrak{p} = \{a \in A \mid a \sim 0\} = \{a \mid 0|a\}$. indeed , if $a \sim 0$ then $0|a$, and conversly, if $0|a$, since anyway $1|0$ (because of (i) and (vi)) with (v) taking $c = a$ we get $a|0$, hence $a \sim 0$.
$\mathfrak{p}$ is a prime ideal : first, if $a, b \in \mathfrak{p}$, $0|a$ and $0|b$ so with (iii) , $0|a+b$, hence $\mathfrak{p}$ stable by +. and with (iv) taking $c = -1$ we have $0|-a$ so $\mathfrak{p}$ is a subgroup. In fact (iii) gives that $\mathfrak{p}$ is an ideal. Now if $a \notin \mathfrak{p}$ and $ab \in mathfrakp$, then $0 \nmid a$ (cf previous remark) , $0|ab$, i.e. $0.a|b.a$ and then with (v) , $0|b$, i.e. $b \in \mathfrak{p}$. So $\mathfrak{p}$ is prime.
Put $B = A/\mathfrak{p}$. Then $|$ factorises through B. Indeed let $a, b \in A$ and $c \in \mathfrak{p}$. If $a|b$, then since anyway $a|0$ and $0|c$ by hypothesis, $a|c$ so (iii) $a|b+c$ hence if $a = a' mod \mathfrak{p}$ we have $a|b$ iff $a'|b$. In particular $a \sim a'$ and we conclude using the transitivity of $|$ that $|$ factorises throug $A/\sim$, and hence also throug B, and that this relation satisfies also $(i)-(vi)$. Actually (v) becomes even ,
$ac|bc$ and $c \neq 0$ implies $a|c$.
Let $K = qf(B)$. Let $x \in K$, with $x = \frac{u}{v} = \frac{u'}{v'}$. Then $v|u$ iff $u'|v'$.
Indeed if $u = 0$ then $u' = 0$ and the two assertions are true.
Otherwise if $v'|u'$ then $v'u|uu'$ but $uv' = u'v$ so $vu'|uu'$ and since $u' \neq 0$, $v|u$.
It then makes sense to define $R = \{x \in K, x = \frac{u}{v} \mid v|u\}$.
This is a valuation ring :
$1 \in R$.
If $\frac{u}{v}$ and $\frac{u'}{v'} \in R$ then $v|u$ hence $vv'|uv'$, $v'|u'$ so $uv'|uu'$ hence by transitivity $vv'|uu'$.
Also $vv'|uv'$ and $vv'|u'v$ so (iii) $vv'|uv' + u'v$, hence $\frac{uv'+u'v}{vv'} = \frac{u}{v} + \frac{u'}{v'} \in R$.
Finally, if $x = \frac{u}{v} \in K$, then by (i) , $u|v$ or $v|u$ so x or $x^{-1} \in R$.
R is then a valuation ring say with $K \xrightarrow{w} \Gamma$ defining its valuation, and if f is the natural morphism $f : A \to B \to K$, then $v = w \circ f$ is a valuation , and by definition of R , if $b \notin \mathfrak{p}$, $a|b$ iff $\frac{\bar{a}}{b} bar0\}$

Remarque 14 *We could consider $\Gamma_0 = (A/\sim, \times)$, check it is an oredred monoid with , $a \geqslant b$ iff $a|b$. Then $\Gamma = \Gamma_0 \backslash \{\bar{0}\}$ would be an orederd submonoid. Then $v : A \to (\Gamma_0, \leqslant)$ is "a valuation in an orderd monoid". So if we could find $(\Gamma, \leqslant) \hookrightarrow (G, \leqslant')$ an injection of orderd monoid with G a group, we could affirm that v comes from a "real" valuation (with value in a group).*
This could lead to consider the forgetfull functor :
for : $Ab \to \{$commutative monoids $\}$, check that it has a right adjoint i defined by $i(M) = (M \cup M^{-1})^*/ < (a).(b) = (ab), aa^{-1} = 1, ab = ba, (ab)^{-1} = (a^{-1}).(b^{-1}) >$. Then wonder if
**the natural morphism $M \to i(M)$ is injective ?*
**Can we extend the oredering of M to $i(M)$?*
It won't be automatic : indeed if $\Gamma = \{-n, -(n-1), \ldots, -1, 0\}$ monoid for $a.b = \max(-n, a+b)$. Then
$v : k[X] \to \Gamma_0$
$P \neq 0 \mapsto -\min(n+1, val_X(P))$
$0 \mapsto -n-1$
and identifying $-(n+1)$ with a null element is a "monoidal" valuation. But $(\Gamma, \leqslant)$ doesnt in an ordered group (it has torsion, and ordered groups don't), v doesn't commes from a valuation : $|v$ verifies $(i)-(iv)$ and (vi) but not (v) :
$0.X|_v X^n.X = X^{n+1}$, $0 \nmid X$, *but however* $0 \nmid X^n$.

We then consider $\phi : S(A) \to \mathcal{P}(A \times A)$ defined by
$\phi(v) = |_v$ with $a|_v b$ if $v(a) \geqslant v(b)$. Then the 6 conditions in the previous proposition show that $im(\phi)$ is a closed set of $\mathcal{P}(A \times A)$, that we endow with the product topology.
Moreover ϕ is injective : indeed if $| = |_v = |_w$ then we easily see that $supp(v) = supp(w) = \{a \in A$ such that $0|a\}$ hence, K the residual field of v and w are the same, and the valuation ring on them induced by v and w are the same (because $|_v = |_w$) so they induce the same valuation on K. Hence through ϕ we identify $S(A)$ with a closed subset of $\mathcal{P}(A \times A)$. It then induces a topology $(S(A), T_1)$. $\mathcal{P}(A \times A)$ being compact, and $S(A)$ closed, $(S(A), T_1)$ is compact. In it the subsets of the form $\{v \mid v(a) \leqslant v(b) \neq 0\} = \{v \mid v(a) \leqslant v(b)\} \cap \{v \mid v(b) \leqslant v(0)\}^c$ are open-closed by definition of the product topology on $\mathcal{P}(A \times A)$. The topology $\mathcal{T}$ they generate is T_0 : if $v \neq w, \in S(A)$ then there exists $a, b \in A$ such that $v(a) \leqslant v(b)$ and $w(a) > w(b)$. If $v(b) \neq 0$ then $v \in \{x | x(a) \leqslant x(b) \neq 0\}$ and not w. Otherwise $v(a) = v(b) = 0$ so $w \in \{x \mid x(b) \leqslant x(a) \neq 0$ and not v.

Lemme 2.1 *Endow $\mathcal{P}(X) = \{0,1\}^X$ with the product topology. Then the closed open subsets are the finite boolean combination of subsets $P_x = \{U \subset X \mid x \in U\}$.*

D : Let V be a closed open subset of $\mathcal{P}(X)$. Since $\mathcal{P}(X)$ is compact (Tychinov) V is compact . Now by definition of the product topology , the P_x and their complementary form a sub-basis of $\mathcal{P}(X)$ so we can conclude.▫

In $S(A)$ the sets $P_{(a,b)}$ correspond precisely to $\{v \mid v(a) \geqslant v(b)\}$. Hence using 2.11 we have :

Proposition 2.21 *$S(A)$ be endowed with the topology whose subbasis is the $\{v \mid v(a) \leqslant v(b) \neq 0\}$. $S(A)$ is spectral and its constructible subspaces are the boolean combination of $\{v \mid v(a) \leqslant v(b)\}$.*

2.7 specializations

Proposition 2.22 (cf [5] 2.2) *Let $v : A \to \Gamma_0$ and H a convex subgroup, $w = v/H : A \to (\Gamma/H)_0$ is called e secondary specialization. $v \in \overline{\{w\}}$ in $Spv(A)$.*

D : let $U = \{x \mid x(f) \leqslant x(g) \neq 0\}$ be a basic neighborhood of v, i.e $v(f) \leqslant v(g) \neq 0$. Then $w(f) \leqslant w(g)$ and $v(g) \neq 0$, i.e. $v(g) \in \Gamma$, so $w(g) \in \Gamma/H$ and is $\neq 0$, so $w \in U$. ▫

exemple : $v : A = k[x,y] \xrightarrow{v} \mathbb{Z}^2_{lex}$
$\sum a_{n,m} x^n y^m \mapsto -\min\{(n,m) | a_{n,m} \neq 0\}$
There are 3 convex subgroups : $\{1\} = \Gamma_0$
$(0, \mathbb{Z}) = \Gamma_1$
$\mathbb{Z} = \Gamma_2$
$c\Gamma_v = \{1\}$, and then :
$v/\Gamma_0 = v$, $v/\gamma_1 = v_x$ (valuation of x).
$v/\Gamma_2 = v_{discret}$.
$v|\Gamma_0 : A \to \{1, 0\}$ with $v|\Gamma_0(f) = 1$ iff $v(f) = 1$, i.e. if $f(0,0) \neq 0$, i.e. it factorises through $A \to k, f \mapsto f(0,0)$, and then with the discrete valuation on k.
$v|\Gamma_1 : A \to \mathbb{Z}_0$,
$f \mapsto v(f)$ if $v(f) \in (0, \mathbb{Z})$, 0 otherwise. factorises through $A \to k[y]$, $f \mapsto f(0,y)$ and then the $y-$adic valuation.
$v|\Gamma_2 = v$.
exemple of the unit ball Let $A = k\{T\}$, $r = |lambda| < 1$, $\lambda \in k$ with k a non-archimedian field. Define :
$\eta_r : \sum a_i T^i \mapsto \max(|a_i| r^i) \subseteq \mathbb{R}$
$\eta_{>r} : \sum a_i T^i \mapsto \max(|a_i| r^i, -i) \subseteq \mathbb{R} \times \mathbb{Z}$
$\eta_{>r} : \sum a_i T^i \mapsto \max(|a_i| r^i, i) \subseteq \mathbb{R} \times \mathbb{Z}$
From what we have seen above since $\eta_{<r}$ and $\eta_{>r}$ are secondary specializations of η_r (with $H = 0 \times \mathbb{Z}$), they both belong to $\overline{\{\eta_r\}}$. The contrary is false (this is a consequence of $Spv(A)$ being T_0) , concretely, if $U = \{x \| \ x(T) \leqslant x(\lambda) \neq 0\}$ then η_r and $\eta_{<r} \in U$, but $\eta_{>r}$ doesnt. In the same way : $V = \{x \| \ x(\lambda) \leqslant x(T) \neq 0\}$ then η_r and $\eta_{>r} \in V$, but $\eta_{<r}$ doesnt. So the specialization described above is the only one existying between these three points.
This shows the difference between the topology of Berkovich and Huber. Indeed if $U = \{v \| \ v(X) <$

$v(\lambda)\}$ then $\eta_{<r} \in U$ and if U was open in the Huber topology, it should contain η_r but this is not the case.

2.8 $c\Gamma_v(I)$

$I = (t_1, \ldots, t_n)$ an ideal.

Lemme 2.2 (2.4) *If $v(I) \cap c\Gamma = \varnothing$ there exists a greatest convex subgroup H such that $v(i)$ is cofinal in H , $\forall i \in I$. Furthermore $v(I) \neq \{0\}$ and $v(I) \cap H \neq \varnothing$.*

D : The existence of H is a consequence of 1. But in this particular case , we have $v(i) < 1 \forall i \in I$.Otherwise $v(i) \geqslant 1$ and then $\in c\Gamma$ by definition of it. Let $h = \max(v(t_j))_{j=1..n} = v(t_1)$ say. If $h = 0$ then $v(I) = 0$ and $H = \Gamma_v$. Otherwise, f $i \in I$, $i = \sum a_k t_k$ and $v(i) \leqslant (\max(v(a_k)).h$ say $v(i) \leqslant v(a)h$. So $v(i^2) \leqslant v(a^2)h^2 = v(a^2 t_1)v(t_1) < v(t_0)$ since $a^2 t_0^i nI$ so $v(a^2 t_0) < 1$. So $v(i)$ is cofinal in $< h >_{conv} =< v(t_1) >_{conv}$. Conversly, if $\forall i \in I$, i is cofinal in H then $v(t_1)$ is cofinal in H and $H \subseteq < h >_{conv}$. So the greatest convex subgroup in which $v(I)$ is cofinal is $H =< v(t_1) >_{conv}$ which then contains $v(t_1) \in v(I)$.▫ $c\Gamma_v(I)$ is then the union of $c\Gamma_v$ and this subgroup H if $v(I) \cap c\Gamma_v = \varnothing$.

Lemme 2.3 (2.5) *If $\Gamma_v \neq c\Gamma_v$ (otherwise $c\Gamma_v(I) = \Gamma_v$). Then the following are equivalent*

1. *$c\Gamma_v(I) = \Gamma_v$*
2. *$v(i)$ is cofinal in Γ_v for all $i \in I$*
3. *$v(i)$ is cofinal in Γ_v for a set of generators of I*

D : 1 ⇒ 2 : since $c\Gamma_v \neq \Gamma_v$, we can't have $v(I) \cap c\Gamma_v \neq \varnothing$, and by definition $v(I)$ is cofinal in Γ_v.
2 ⇒ 1 : then $v(I) \cap c\Gamma_v = \varnothing$. Otherwise if $v(i)\ inc\Gamma_v$, $g < c\Gamma_v$, then there exists a n such that $v(i)^n < g < c\Gamma_v$ which is absurd. Hence $v(I) \cap c\Gamma_v = \varnothing$ and $c\Gamma_v(I) = \Gamma_v$.
2 ⇒ 3 is clear.
3 ⇒ 2 : The set $J = \{a \in A \mid v(a)$ is cofinal in $\Gamma_v\}$ is an ideal. Indeed first $v(J) < 1$, and if $g \in \Gamma_v$, $a, b \in J$, $\exists n$ such that $v(a^n) < g$ and $v(b^n) < g$ then $v((a+b)^{2n}) < g$. If $x \in A$, then if $v(x) leq1$, $v(ax) \leqslant v(a)$ and $ax \in J$. Otherwise, $v(x) \geqslant 1$, then $v(x) \in c\Gamma_v$, $v(ax) \leqslant v(x)$. Now if $1 \leqslant v(ax)$ we have $v(ax) \in c\Gamma_v$ and $v(a)$ too, which is impossible since $c\Gamma_v \neq \Gamma_v$ and $v(x)$ is cofinal. Hence $v(ax) < 1$ for all $x \in A$. Let then $g \in \Gamma_v$. There exists n such that $v(a)^n < g$ then $v(ax)^{n+1} = v(a^n)v(ax^{n+1}) < v(a^n) < g$. So J is an ideal, and 3 ⇒ 2.▫

Remarque 15 *If $I = A$ then $v(I) \cap C\Gamma_v \neq \varnothing$ so $c\Gamma_v(I) = c\Gamma_v$, and then $Spv(A, A) = \{v \mid c\Gamma_v = \Gamma_v\}$.*

3 Continuous valuation of f-adic rings

Remarque 16 *Let A be a Tate ring, and $v : A \to \Gamma = \Gamma_v$ be a continuous valuation. Then $c\Gamma = \Gamma$. Indeed take x a nilpotent unit. Then x is cofinal in Γ , and $< x >_{conv} = \Gamma$. It is even true that the subgroup generated by $\{v(a) \geqslant 1\}$ is Γ. So in that case there are only secondary specializations.*

Theorem 1 (3.1) *$Cont(A) = \{v \in Spv(A, A.A^{\circ\circ}) \mid v(A^{\circ\circ}) < 1\}$.*

D : If v is continuous. Then clearly $v(A^{\circ\circ}) < 1$. Then, if $c\Gamma_v = \Gamma_v$ OK. Otherwise let $A \in A^{\circ\circ}$, $v(a^n) \to 0$ so $v(a)$ is cofinal in Γ_v and according to 2.8, $v \in Spv(A, A.A^{\circ\circ})$, i.e. $c\Gamma_v(A.A^{\circ\circ}) = \Gamma_v$.
Conversly let $v \in Spv(A, A.A^{\circ\circ})$ such that $v(A^{\circ\circ}) < 1$. First let's show that $\forall a \in A^{\circ\circ}$, $v(a)$ is cofinal in Γ_v. If $\Gamma_v \neq c\Gamma_v$ then this is true by definition of $Spv(A, A.A^{\circ\circ})$ and $c\Gamma_v(A.A^{\circ\circ})$. Otherwise $\Gamma_v = c\Gamma_v$, hence if $g \in \Gamma_v$ $\exists t \in A$ such that $v(t) \geqslant v(g)^{-1}$ i.e. $v(g) \geqslant v(t)^{-1}$. Hence if $A \in A^{\circ\circ}$, $\exists n$ such that $v(a^n t) < 1 \Rightarrow v(a^n) < g$.
So let $A_0, I = (b_1 \ldots b_n)$ be an adic ring of definition for A. Since the $b_i \in A^{\circ\circ}$, the $v(b_i)$ are cofinal from what we've just seen, in particular $v(b_i) < 1$, and we easily see that for $\nu = (k_1, \ldots k_n)$ with $|\nu| \geqslant N$ for a big enough N , $v(b^\nu) < g$. Hence since $v(I) < 1$ we have $v(I^{N+1}) < g$ which shows v is continuous. ▫

Theorem 2 *If A is a ring which is Tate, $A.A^{\circ\circ} = A$ and hence $Cont(A) = \{v \in Spv(A) \mid c\Gamma_v = \Gamma_v$ and $v(A^{\circ\circ}) < 1$*

3.1 counter-example continuous valuations

1. $v(A^{\circ}) \leqslant 1$, $v(A^{\circ\circ}) < 1$: inspired by [2] §10 lemma 1 which says that if $v : k \to \Gamma_0$ is a valuation of the field k and $g \in \Gamma$ $w : k[X] \to \Gamma$, $\sum a_i X^i \mapsto \max(v(a_i)g^i)$ is a valuation.
   Let $v : A = k\{X\} \to \mathbb{Z} \times \mathbb{R}$, $\sum a_n X^n \mapsto \max(-n, |a_n|)$. This is a valuation, $v(A^{\circ\circ}) < 1$ and $v(A^{\circ}) \leqslant 1$ but it is not continuous : $c\Gamma_v = 0 \times \mathbb{Z}$. Or if $\pi \in k^{\circ}$ $v(\pi)^n$ isn't aribtrary small although $\pi^n \to 0$. Here $v \in L(A) \backslash Cont(A)$.
2. *A Tate ring and a v such that $v(A^{\circ\circ}) < 1$, $v(A^{\circ}) \leqslant 1$ but not continuous.* Take $A = \mathbb{Z}_p[X, X^{-1}] \supseteq A_0 = \mathbb{Z}_p[X] \supseteq I = A_0.X$. The X-adic toplogy on A_0 extended to A makes it a f-adic ring. Mainly because if $P \in A$, $f_n \to 0$ then $Pf_n \to 0$ and if $g_n \to 0$, $f_n g_n \to 0$. Then $A^{\circ\circ} = I$; $A^{\circ} = A_0$. Let $v : A \to \mathbb{Z}^2$, $v(\sum a_i X^i) = \max(|a_i|, -i)$. Then $v(I) < 1$, $v(A^{\circ}) \leqslant 1$ but $X^n \to 0$ however $v(X^n)$ doesn't converges to 0 , i.e. $v(X)$ isnot cofinal in Γ_v. (Also because $c\Gamma_v = 1 \times \mathbb{Z} \neq \mathbb{Z}^2$.)
3. A a f-adic ring , $v : A \to \Gamma_0$ such that $v(A^{\circ\circ}) < 1$ but v not continuous.
   Take $A = \mathbb{Z}_p[X]$ equiped with the (X,p)-adic toplogy, and $v : A \to \mathbb{Z}^2$, $\sum a_i X^i \mapsto \max(|a_i|_p, -i)$. Then on easily checks that $A^{\circ\circ} = (p, X)$ and that $v(A^{\circ\circ}) < 1$. But $X^n \to 0$ however $v(X^n) = (1, -n)$ doesn't converge to 0.

Proposition 3.1 (cf [5] 3) *The integral closure of the subring $\mathbb{Z} + A^{\circ\circ} = \{n + a \mid n \in \mathbb{Z}, a \in A^{\circ\circ}\}$, B is the smallest ring of integral elements of A.*

D : First note that $A^{\circ\circ}$ is open. Indeed if (A', I) , is an adic ring of definition of A , $I \subset A^{\circ\circ}$ is open. So $\mathbb{Z} + A^{\circ\circ}$ is a subring of A° (note that $\mathbb{Z} + A^{\circ\circ}$ is well a subring, because $A^{\circ\circ}$ is stable by + and $\times$).
Now let's show that A° is integrally closed in A.
First let's prove :

Lemme 3.1 *If B is a bounded subring in A f-adic, and $A \in A^{\circ}$ then $B[a]$ is bounded.*

D : let (A_0, I) be a ring of definition. I^n a neighborhood. $\exists m$ such that $\{a^k\} I^m \subseteq I^n$, and p such that $BI^p \subseteq I^m$. Then $ba^n I^p \subseteq a^n I^m \subseteq I^n$. ▫. Hence since $\mathbb{Z}$ is also bounded, we see that if $a_0, \dots a_n \in A^{\circ}$, $\mathbb{Z}[a_0, \dots, a_n]$ is also bounded.
So let $x \in A$ be integral on A°, i.e. $x^n = a_0 + a_1 x + \dots a_{n-1} x^{n-1}$. Call $B = \mathbb{Z}[a_0 \dots a_{n-1}]$. By induction we got that $x^p \in B + B.x + \dots + B.x^{n-1}$ $\forall p$. So if I^m is a 0 neighborhood, we can find k such that $B.x^l I^k \subseteq I^m$ for all $l = 0..n-1$. Then $x^p I^k \subseteq I^m$, $\forall p$. Hence $x \in A^{\circ}$. So A° is integrally closed in A , so $(\mathbb{Z} + A^{\circ\circ})^{closure\ A} \subseteq A^{\circ}$.
Conversly, if $B \subseteq A^{\circ}$ is open and integrally closed in A, let $x \in A^{\circ\circ}$, so that $x^n \to 0$, hence there exists a n such that $x^n \in B$ since it is open, hence $x \in B$ since it is integrally closed. ▫

[3.6] supposes that $\{\bar{0}\}$ is an ideal : let $x \in \bar{0}$, $a \in A$ V a 0-neigborhhod. We can assume $V = -V$, then $0 \in ax + V \Leftrightarrow ax \in V$, but $A \to A$, $u \mapsto au$ is continuous so $\exists$ a neigborhood of 0 , W such that $aW \subseteq V$, and $x \in W$ because $x - W$ is a neighborhood of x so $0 \in x - W$, i.e. $x \in W$. ▫

3.2 Affinoid rings

Définition 3.1 *A subring of A is called a ring of integral elements if it is open , integraly closed, and contained in A°.*
An affinoid ring is a pair (A, A^+) with A a f-adic ring and A^+ a subring of integral elements.
By a ring homomorphism of affinoid ring it is meant f such that $f(A^+) \subseteq B^+$.

Lemme 3.2 *Let J be an ideal of A. J is open $\Leftrightarrow$ $A^{\circ\circ} \subseteq \sqrt{J}$*

D : $\Rightarrow$ Let $a \in A^{\circ\circ}$, so that $\exists n$ such that $a^n \in J$.
$\Leftarrow$ Let (A_0, I) be a ring of definition, $I = (i_1, \dots, i_n)$. Since $I \subseteq A^{\circ\circ} \subseteq \sqrt{J}$, for each $j = 1..n$ there exists k_j such that $i_j^{k_j} \in J$. Then $k := \sum k_j$ and $I^k \subseteq J$ which is then open. ▫

This explains that if $T.A$ is open, $A^{\circ\circ} \subseteq \sqrt{J}$ so $U = \{v \in Spv(A, A.A^{\circ\circ}) \mid v(t_i) \leqslant v(t) \neq 0\}$ a rationnal subset.

Remarque 17 *On rational domains. Let's restrict to the case A a ring Tate. Then for [5] a rationnal subset of $Spa(A)$ is $\mathcal{R}(\frac{T}{s}) = R = \{v \mid v(t_i) \leqslant v(s) \neq 0\}$ with $(t_i) = A$. Hence if $v \in \{v \mid v(t_i) \leqslant v(s)\}R$ since $\sum a_i t_i = 1$ if $v(s) = 0$ we have $v(t_i) = 0$ so $v(1) = 0$ which is impossible, so $R = \{v \mid v(t_i) \leqslant v(s)\}$.*
Now let's consider $S = \{v \mid v(t_i) \leqslant v(s), i = 1..n\}$ where $(t_i, s) = A$. If we had $t_{n+1} = s$ we still have $S = \{v \mid v(t_i) \leqslant v(s), i = 1..n+1\}$ which is rationnal in Huber's sense. So the two possible definition of $\mathcal{R}(\frac{T}{s})$, with $(T) = A$ or $(T, s) = A$ give the same class of subsets.

3.3 analytic points

There exists a finite set $T \subseteq A^{\circ\circ}$ such that $T.A$ is open. Indeed let (B, I) a ring of definition , with $I = (b_1, \ldots b_n)$. Then $T = \{b_1, \ldots, b_n\}$ works.

Proposition 3.2 $(Spa(A))_a = \{x \mid supp(x) = v^{-1}(0)$ *is not open* $\}$
$= \{x \mid x(t) \neq 0$ *for one* $t \in T\}$
$= \cup_{t \in T} R(\frac{T}{t})$

D : if $x \in (Spa(A))_a$ then $supp(x)$ is an ideal, not open, so $T \nsubseteq supp(x)$ so $\exists \mid x(t) \neq 0$.
Conversly, if $x(t) \neq 0$, $t \in A^{\circ\circ}$, $x(t^n) \neq 0$ but converges to 0 so $supp(x)$ is not open. □

Proposition 3.3 *If $v \in Spa(A)$ is analytic, then v is microbial.*

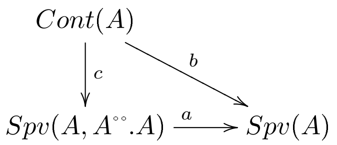
D : We have $A \to A/supp(v) = B \to qf(B) = K$ and let R be the valuation ring of K associated to v.

Let also A_0, I be a ring of definition of A. First note that saying that v is analytic means that $supp(v)$ is not open (it is closed, but we don't care), so $I \subsetneq supp(v)$ and in fact neither I^n for any n. Then the topology on R is I-adic (more precisely we should define $J = R.\{\bar{i} \mid i \in I\}$ and say R is J-adic. Indeed, first J is an ideal of R wich is not $\{0\}$ (because $I \subsetneq supp(v)$) , so it is open. And if $g \in \Gamma_v$, we have a n such that $v(I^n) < g$. Then $v(J^n) < g$ which shows that the topology of R is the J-adic topology. Since $J^n \neq \{0\}$ for all n , the topology is not discrete. So accoridin to the criterion 4 , v is microbial. □

3.4 constructible sets in the respective spaces

In $Spv(A)$, the constructible subsets are finite boolean combination of subsets of the form $\{v \mid v(a) \leqslant v(b)\}$ (prop 2.2 in [5]). This includes for instance the subsets $\{v \mid v(a) = v(b)\}$ and $\{v \| \ v(a) = 0\}$.

For instance $U = \{v \in Spv(A) \mid v(a) \neq 0\}$ is quasi compact open (i.e. constructible and open, i.e. proconstructible and open) , because indeed it is open, $(= \{v | v(a) \leqslant v(a) \neq 0\}$ and constructible.

In $Spv(A, I)$ the constructible subsets are the boolean combinations (finite) of rational domains :
$U = R(\frac{T}{s}) = \{v \in Spv(A, I) \mid v(t) \leqslant v(s) \forall t \in T\}$ where T is finite and $I \subseteq \sqrt{T.A}$.
$Cont(A) = \{v \in Spv(A, A.A^{\circ\circ}) | v(A^{\circ\circ}) < 1\}$ is a proconstructible subset of $Spv(A, A^{\circ\circ}.A)$.
Let's consider 

Then a and b are not spectral in general.

Indeed otherwise $b^{-1}\{v(a) \leqslant v(b)\}$ would be constructible, but it is hard to imagine how it could be a boolean combination of rationnal subset , particularly when $b = 0$ in which case it is a Zariski closed subset. More precisely if A is an affinoid algebra , $f \neq 0$, $\pi \in k^{\circ\circ} \backslash \{0\}$, then $b^{-1}(v(f) \neq 0) = \cup_{n \geqslant 0}\{v(f) \geqslant v(\pi^n)\}$ and you can't extract a finite cover from the right hand side, so it is not quasi compact. So b isn't spectral , and since $b = c \circ a$, and a is spectral, c isn't spectral.
according to [5, 2.5 , 2.6] :
$r : Spv(A) \to Spv(A, I)$
$v \mapsto v|c\Gamma_v(I)$ is spectral. Then take $A = k[t]$, $I = A$
It is false to say that $r^{-1}\{v \in Spv(A, A) \mid v(a) \leqslant v(b)\} = \{v \in Spv(A) \mid v(a) \leqslant v(b)\}$.
For instance $U = r^{-1}\{v \mid v(T) \geqslant 1\}$. Let v_T be the $T-$adic valuation. $c\Gamma_{v_T} = \{1\}$ so $c\Gamma_{v_T} = \{0\}$

and $v_T|\{0\} = w =$ the trivial valuation . Hence since $w(T) \geqslant 1$, $v_T \in U = r^{-1}\{v \mid v(T) \geqslant 1\}$ but $v_T \notin \{v \in Spv(A) \mid v(T) \geqslant 1\}$ so $r^{-1}\{v \mid v(T) \geqslant 1\} \neq \{v \in Spv(A) \mid v(T) \geqslant 1\}$.
This helps to understand the fact that :
$r : Spv(A) \to Spv(A, I)$ is spectral [5, 2.6(ii)] , and surjective, so $T \subseteq Spv(A, I)$ is constructible iff $r^{-1}(T)$ is . So if it was true that $r^{-1}\{v \in Spv(A, I) \mid v(a) \leqslant v(b)\} = \{v \in Spv(A) \mid v(a) \leqslant v(b)\}$, the subsets $\{v \in Spv(A, I) \mid v(a) \leqslant v(b)\}$ would be constructible.

4 Tate rings of topologically finite type over fields

Proposition 4.1 *Let $\mathcal{A}$ be a k-affinoid algebra, and $v \in Spa(\mathcal{A}, \mathcal{A}^{\circ})$. Then $v_{|k}$ is the initial valuation of k.*

Indeed, v is a valuation with $v(x) \geqslant 1$ when $x \in \mathcal{A}^{\circ} \cap k = k^{\circ}$. Conversly if $x \notin \mathcal{A}^{\circ} \cap k$, then $x^{-1} \in \mathcal{A}^{\circ\circ}$ so $v(x^{-1}) < 1$ (cf 3.1 [5]), i.e. $v(x) > 1$. So $v(x) \leqslant v(y)$ iff $|x| \leqslant |y|$ so they are the same.
More conceptually $(k, k^{\circ}) \to (\mathcal{A}, \mathcal{A}^{\circ})$ is a continuous morphism of affinoid ring, so induces $f : Spa(\mathcal{A}, \mathcal{A}^{\circ}) \to Spa(k, k^{\circ})$. What we've shown is somehow the fact that $Spa(k, k^{\circ}) = \{||\}$ where $||$ is the valuation on k, because in general (cf [6] 1.1.6) if $A = (A^{\rhd}, A^{+})$ is an affinoid fiels, $Spa(A)$ is the set of valuation ring B, such that $A^{+} \subseteq B \subseteq (A^{\rhd})^{\circ}$. In our case it gives $k^{\circ} \subseteq B \subseteq k^{\circ}$, so the only possiblity is k°.□

If A is a Tate algebra, $L_A = \{v \in Spv(A \mid v(A^{\circ}) \leqslant 1$ and $v(A^{\circ\circ}) < 1\}$. We note (abuse of notation) $Spa(A) := Spa(A, A^{\circ})$. Then

$$Max(A) \subseteq Spa(A) \begin{array}{c} \subseteq Cont(A) \subseteq \\ \\ \subseteq L_A \subseteq \end{array} Spv(A)$$

Proposition 4.2 ([8] the. 10.2) *Let K be a field , $A \subseteq K$ a subring $\mathfrak{p}$ a prime ideal of A. Then there exists a valuation ring R of K such that $A \subseteq R$ and $\mathfrak{M}_R \cap A \subseteq \mathfrak{p}$.*

Corollaire 4.1 *Let k be a valued field and K an extension of field, then there exists a valuation on K that extends the one of k.*

D : Let A be the valuation ring of (k, v), $\mathfrak{p} = \mathfrak{m}_A$. Then there exists R a valuation ring of K with $A \subseteq R$ and $\mathfrak{m}_R \cap A \subseteq \mathfrak{m}_A$. So let $C = k \cap R$. Then $C = A$ (for instance because C is a k-valuation ring that extends A, with the same maximal ideal, or because if $x \in C \backslash A$, $v(x) > 1$, $x^{-1} \in \mathfrak{m}_A$ but since $\mathfrak{m}_R \supseteq \mathfrak{m}_A$, $x^{-1} \in \mathfrak{m}_R$ which contradicts $x \in R$. □

Proposition 4.3 *Let A be a ring, $I = (a_j)_{j \in J}$ an ideal. Then $\pi : A \to A/I$ induces $Spv(\pi) : Spv(A/i) \to Spv(A)$. Its iamge is $\{v \mid v(I) = 0\}$ and it is a homeomorphism on its image. In particular, if I is of finite type, this image is a constructible subset.*

Proposition 4.4 (cf [5] 4.1 or [7] Prop. 2.1.1) *Let $f : A \to B$ a morphism of finite presentation and $U \subseteq Spv(B)$ a constructible subset. Then $Spv(f)(U)$ is constructible .*

D : f decomposes as $A \xrightarrow{f_1} A[X_1, \ldots X_n] \xrightarrow{f_2} B = A[X_1, \ldots, X_n]/I$ where $I = (a_1, \ldots, a_n)$ is finitely generated. Let U be a boolean combination of $\{v(\bar{a}) \Diamond v(\bar{b})\}$ with $a, b \in A[X_1 \ldots X_n]$, then $Spv(f_2)$ is the same boolean combination of $\{v(a) \Diamond v(b)\} \cap \{v(a_i) = 0,\ i = 1 \ldots n\}$.
So we can restrict to the case $B = A[X_1, \ldots, X_n]$, U a boolean combination of $\{v(P) < v(Q)\}$. Since $P \in A[X_1, \ldots, X_n]$, $\exists$ an interger m, $p_1, \ldots p_m \in A$ and $p \in \mathbb{Z}[Y_1, \ldots, Y_m, X_1, \ldots X_m]$ such that $P = p(p_1, \ldots, p_m, X_1, X_n)$, and also $Q = q(q_1, \ldots, q_M, X_1, \ldots, X_n)$.
An element $w \in Spv(A[X_i]$ represented by $B \xrightarrow{\psi} k$ is in U iff the combination of formula $p(p_i, t_j) < q(q_k, t_j)$ is true where $t_j = \psi(X_j)$.
Hence $A \xrightarrow{\phi} k$ corresponds to a valuation v of A, it is in $Spv(f)(U)$ iff there exists a diagram

$$\begin{array}{ccc} B & \xrightarrow{\psi} & L \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \iota} \\ A & \xrightarrow{\phi} & k \end{array}$$

But when ϕ and ι are fixed, a ψ giving rise to a commutative diagram as this one is equivalent to the data of $l_1, \ldots l_n \in L$.
Hence $v \in Spv(f)(U)$ iff
$\exists$ an extension L of k and $l_1, \ldots l_n \in L$ such that the formula $P(l_1, \ldots, l_n) < Q(l_1, \ldots, l_n)$ is true iff $\exists \iota : k \to L$ an extension with L algebraically closed valued field (using 4.1), and such that the following formula holds

$$\exists l_1 \ldots l_n \text{boolean combination}(|p(p_i, l_j)| < |q(q_k, l_j))$$

For such a L, if it is trivially valued, we can embed it in $L(X)$ with the X-adic valuation so that it isn't trivially valued anymore so that
$\Leftrightarrow \exists \iota : k \to L$ an extension with L algebraically closed non-trivially valued field (using 4.1), and such that the following formula holds

$$\exists l_1 \ldots l_n \text{boolean combination}(|p(p_i, l_j)| < |q(q_k, l_j))$$

But using elimination of quantifiers for the non-trivially valued fields (warning you can't eliminate $\exists x \neq y \neq 0 |x| \neq |y|$, which precisely defines the non trivially valued fields), this formula is equivalent to a universal (meaning independant of L) formula $\varphi(p_i, q_k)$, which defines a constructible subset of $Spv(A)$.□

Theorem 3 (4.1) *L_A is the closure of $Max(A)$ in the constructible topology of $Spv(A)$*

D : First $L(A)$ is well closed in this topology (cf Prop 2.2) wich says that a basis for the constructible topology is the sets $\{v \mid v(a) \diamond v(b)\}$, $\diamond \in \{<, \leqslant\}$. □

Proposition 4.5 *Let $f : X \to Y$ a continous map beetwen toplogical spaces, $A \subseteq Y$.*

1. $\bar{f^{-1}}(A) \subseteq f^{-1}(\bar{A})$
2. *if f is open* $\bar{f^{-1}}(A) = f^{-1}(\bar{A})$

D : $1 f^{-1}(\bar{A})$ is closed and contains $f^{-1}(A)$.
2 Let $x \in f^{-1}(\bar{A})$ and U a neigborhhod of x. We have to show that $U \cap f^{-1}(A) \neq \emptyset$. But $f(U)$ is open, so neigborhhod of $y = f(x) \in \bar{A}$, so $f(U) \cap A \neq \emptyset$. So if $z \in f(U) \cap A$, $z = f(u)$, $u \in f^{-1}(A) \cap f^{-1}(f(u)) \subseteq f^{-1}(A) \cap U$, $\Rightarrow$ $f^{-1}(A) \Cap U \neq \emptyset$. □

4.1 Prime filters

$\tilde{Max}(A)$ denotes the set of prime filters of $Max(A)$, (precisely the prime filters of the lattice of finite union of rationnal subsets (cf Dickmann).
Cor 4.5 : Let $\mathcal{F}$ be a prime filter, define $\mathcal{F}' = \{Max(A) \backslash R \mid R \notin \mathcal{F}\}$ and define $\mathcal{W} = \mathcal{F} \cup \mathcal{F}'$.

$$\| \text{Let } W_1, \ldots, W_n \in \mathcal{W}, \text{ then } \cap_{i=1..n} W_i \neq \emptyset$$

D : in this intersection there is in fact one rational domain R (because they are stable par $\cap$) , and some R_i^c with $R_i \notin \mathcal{F}$. Then if we had $R \subseteq \cup_i R_i$, $R = \cup(R_i \cap R)$ is an element of $\mathcal{F}$ so one of the $R \cap R_i$ must also be in, so R_i also which is absurd. So $R \subsetneq \cup_i R_i$, i.e. $R \cap_i R_i^c \neq \emptyset$. □

$$\| D := \cap_{W \in \mathcal{W}} \tilde{W} \neq \emptyset. \text{ Let } \ x \in D, \text{ then } s(x) = \mathcal{F}$$

D : $s(x) \supseteq \mathcal{F}$: if $F \in \mathcal{F}$, $x \in D$ so $x \in \tilde{F}$.
Conversly let $\tilde{U} \cap Max(A) = U \in s(x)$, i.e. $x \in \tilde{U}$. If we had $U \notin \mathcal{F}$, then $V = Max(A) \backslash F \in \mathcal{W}$, and then $x \in \tilde{V}_2$ so $x \notin \tilde{U}$, absurd.□
4.7.2 : $\mathcal{F} \in \tilde{Max}(A)$, then :
$1\{a \in A \mid \forall F \in \mathcal{F} \exists x \in F \mid a(x) = 0\}$
$2 = \{a \mid \forall e \in k^* \exists F \in \mathcal{F} \mid |a|_F \leqslant |e|\}$
$3 = \{a \mid \forall r > 0 \ \exists F \in \mathcal{F} \mid |a|_F \leqslant r\} = \mathfrak{p}_{\mathcal{F}}$
2 =3 is clear.

$1 \subseteq 3$: if $a \in 1$. Let $F_1 = \{x \in Max(A) \mid |a(x)| \leqslant r\}$ and $F_2 = \{x \in Max(A) \mid |a(x)| \geqslant r\}$. $F_1 \cup F_2 = Max(A)$ so one of it is in $\mathcal{F}$. $F_2 \notin \mathcal{F}$ because $a \in 1$, and $\forall x \in F_2$ $a(x) \neq 0$. So $F_1 \in \mathcal{F}$, and $a \in 3$.
If $a \in 3$. Let $F \in \mathcal{F}$ such that $a(x) \neq 0 \forall x \in F$. Then $\exists r >$ such that $|a(x)| \geqslant 2r$ $\forall x \in F$. But since $a \in 3$ $\exists G \in \mathcal{F}$ such that $|a|_G \leqslant r$. Then $F \cap G \in \mathcal{F}$, but is empty. Contradiction, so $\exists x \in F$ such that $a(x) = 0$.
Remark : with 3 , we see that $\mathfrak{p}_{\mathcal{F}}$ is prime ideal. Indeed if $a, b \in 3$ and $r > 0$, $\exists F_a, F_b$ such that $|a|_{F_a} \leqslant r|$... Then $|a+b|_{F_a \cap F_b} \leqslant r$. If $c \in A$, $|ac|_{F_a} \leqslant \|c\||a|_{F_a} \leqslant \|c\|r$. And if $a, b \in A$ and $ab \in 3$. Let $r > 0$, F such that $|ab|_f \leqslant r^2$, $F_a = \{x \mid |a(x)| \leqslant r\}$, $F_b = \{x \mid |b(x)| \leqslant r\}$. Then $F_a \cup F_b \supseteq F$, so one of them is in $\mathcal{F}$.
Rk : we prooved that $\mathfrak{p}_{\mathcal{F}}^c = \{a \mid \exists F \in \mathcal{F}, r > 0 \ |\forall x \in F|a(x)| \geqslant r\}$.
$s(\eta_1) = \{R \mid R$ contains all but finitely many open balls of radius $1\}$.
$\subseteq s(\eta_{<1}) = \{R \mid R \supseteq (\mathring{B}(0,1)$ minus some balls od radius$< 1\}$.

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