Complex semi-algebraic subsets are not stable under projection

Quantifier elimination in field theories, is another way to say that definable (or constructible, semi-algebraic) subsets are stable under projection, or say algebraic maps.
This is true for constructible subset of K^n when K is an algebraically closed field, and is known as Chevalley theorem in the geometric side [Har00][ex. II-3.19] , and the fact that the theory of algebraically closed fields has quantifier elimination in the model theoretic side [Mar00][3.2.2].
This is true in the framework of real algebraic geometry , and is known as Tarski Theorem which states that semi-algebraic subsets are stable under projection[BCR98].
This is also true in the framework of algebraically closed valued fields [Rob77] or [Duc03] for a more geometric approach.
It is then quite natural to wonder what happens for the field $\mathbb{C}$ equiped with its absolute value $|.|$. We show that in this framework, *complex* semi-algebraic subsets are not stable under projection.

Recall the following definition :

Definition 0.1. *A subset $V \subseteq \mathbb{R}^n$ is called a semi-algebraic subset if V is a finite boolean combination of subsets of the form $\{x \in \mathbb{R}^n \ |f(x) > 0\}$ where $f \in \mathbb{R}[x_1, \ldots, x_n]$, and by finite boolean combination, we mean using finitely many times the symbols $\cap$, $\cup$ and ^{c}.*

Definition 0.2. *A subset $V \subseteq \mathbb{C}^n$ is called a complex semi-algebraic subset if V is a finite boolean combination of subsets of the form $\{z \in \mathbb{C}^n \mid |f(z)| < |g(z)|\}$ where $f, g \in \mathbb{C}[z_1, \ldots, z_n]$.*

In these definitions, we could also allow subsets defined with $\leq$ and $=$, since they can be obtained from $>$ and the boolean operators $^c, \cap$.
If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, one sees that a complex semi-algebraic subset of $\mathbb{C}^n$ is a semi-algebraic subset of $\mathbb{R}^{2n}$. Lemma 0.1 and 0.2 will clarify in what sence the converse is true or false.

Remark 1. *Let $V = \{z \in \mathbb{C}^n \mid |f(z)| = |g(z)|\}$. Then we are in one of the three cases :*

1. *f and g are constant, in which case $V = \mathbb{C}^n$ or is empty.*

2. *There exists a $\lambda \in \mathbb{C}^*$ such that $g = \lambda f$, in which case V is equal to $\mathbb{C}^n$ if $|\lambda| = 1$ and is empty otherwise.*

3. *In the other cases V is a strict Zariski-closed subset of $\mathbb{R}^{2n}$ (and in fact non empty : after factorization of the polynomials, dividing by their common factors, we can assume they have no common factor, and are not each of them constant -oterwise we would be in case 2- then if one of them is constant λ, since the otherwise is not, it vanishes, and tends to infinity, so by mean value theorem reaches $|\lambda|$ at some point, othermise they both vanish at different points, and by the mean value theorem we conclude again).*

We'll call the third case a strict equality , and since the two other cases lead to trivial complex semi-algebraic subset , we'll assume that all equality appearing are strict equalities.

It is easily seen that a complex semi-algebraic subset can been writen as a finite union of sets of the form :

$$\left(\bigcap_{i=1}^{n}\{z \in \mathbb{C}^n \mid |f_i(z)| < |g_i(z)|\}\right) \bigcap \left(\bigcap_{j=1}^{m}\{z \in \mathbb{C}^n \mid |F_j(z)| = |G_j(z)|\}\right)$$

with $f_i, g_i, F_j, G_j \in \mathbb{C}[z_1, \dots, z_n]$, and m (resp. n) being possibly zero, which would mean that there would be only $=$ (resp. $<$) . We'll call such an intersection a *basic complex semi-algebraic subset* (the equalities here are assumed to be strict).

Remark 2. *In the definition of (real) semi-algebraic subset of $\mathbb{R}^n$, if we replace inequalities $f > 0$ by $|g| > |h|$ (with g, h real polynomials) we get the same definition.*
Indeed in one hand $|g| > |h|$ is equivalent to $g^2 - h^2 > 0$. One the other hand, $f > 0$ is equivalent to $|f + 1| > |f - 1|$.

Lemma 0.1. *Let V be a (real) semi-algebraic subset of $\mathbb{C}^n$ (we mean by that a semi-algebraic subset of $\mathbb{R}^{2n}$). Then there exists $m \in \mathbb{N}$, W a complex semi-algebraic subset of $\mathbb{C}^m$ such that $V = \varphi(W)$ where φ is a complex polynomial map $\varphi : \mathbb{C}^m \to \mathbb{C}^n$.*

Proof. Let us note that in $\mathbb{C}$, $\mathbb{R} = \{z \in \mathbb{C} \mid |z - i| = |z + i|\}$. We then take $m = 2n$, and writing the k-th complex coordinate of $\mathbb{C}^n$ $z_k = x_k + iy_k$, let us suppose that $f(x_1, \dots, x_n, y_1, \dots, y_n)$) and $g(x_1, \dots y_n)$ are two polynomials (in 2n real variables) such that $V = \{(x_1 + iy_1, \dots x_n + iy_n) \in \mathbb{C}^n \mid |f(x_1, \dots, x_n, y_1, \dots, y_n)| < |g(x_1, \dots, y_n)|\}$. We now consider $\mathbb{C}^{2n}$ with the complex coordinates $X_1, \dots, X_n, Y_1, \dots, Y_n$. Then we define

$$W_1 = \{(X_1, \dots, X_n, Y_1, \dots, Y_n) \in \mathbb{C}^{2n} \mid |X_k - i| = |X_k + i| \text{ and } |Y_k - i| = |Y_k + i| \text{ , } k = 1 \dots n\}$$

Then by the previous remark W_1 is the set of points of $\mathbb{C}^{2n}$ with real coordinates. Set $W_2 = \{(X_1, \ldots, Y_n) \in \mathbb{C}^{2n} \mid |f(X_1, \ldots, Y_n)| < |g(X_1, \ldots, Y_n)|\}$, and $W = W_1 \cap W_2$. Finally let

$$\begin{array}{cccc} \varphi : & \mathbb{C}^{2n} & \to & \mathbb{C}^n \\ & (X_1, \ldots, X_n, Y_1, \ldots, Y_n) & \mapsto & (X_1 + iY_1, \ldots, X_n + iY_n) \end{array}$$

Then $\varphi(W) = V$. Then using remark 2 and making use of finite boolean combination makes it work. □

Lemma 0.2. *Let $\mathcal{H} = \{x + iy \in \mathbb{C} \mid y^2 - (\cos(1)x)^2 = 1\}$. Then $\mathcal{H}$ is not a complex semi-algebraic subset of $\mathbb{C}$.*

Proof. Otherwise let us write $\mathcal{H} = \cup_{k=1\ldots N} V_k$ with the V_k basic complex semi-algebraic subsets. One of the V_k , let us call it V, must contain inifinitely many points of $\mathcal{H}$. Write

$$V = \left(\bigcap_{i=1}^{n} \{z \in \mathbb{C} \mid |F_i(z)| < |G_i(z)|\} \right) \bigcap \left(\bigcap_{j=1}^{m} \{z \in \mathbb{C} \mid |f_j(z)| = |g_j(z)|\} \right)$$

V can't be open, since it is contained in $\mathcal{H}$ whose interior is empty, so in the expression of V there is indeed an equality $|f(z)| = |g(z)|$ (i.e. $m \geq 1$) , which defines a stric Zariski-closed subset of $\mathbb{C} \simeq \mathbb{R}^2$. If we call $W = \{z \in \mathbb{C} \mid |f(z)| = |g(z)|\}$, by assumption, W is a strict (real) Zariski-closed subset of $\mathbb{C}$ which contains infinitely many points of $\mathcal{H}$, so $\mathcal{H} \subseteq W$ (because $\mathcal{H}$ is irreducible). Since $\frac{f}{g}$ is not constant , it has an assymptotic taylor serie of the form $\frac{f(z)}{g(z)} = 1 + \frac{\alpha}{z^p} + \circ(\frac{1}{|z^p|})$ with $\alpha \neq 0$ and $p > 0$, and by assumption, for all $z \in \mathcal{H}$ where g doesn't vanish , $|\frac{f(z)}{g(z)}|^2 = 1$. When passing to the absolute value , the asymptotic serie gives us for $z \in \mathcal{H}$, $1 + 2\Re(\alpha z^{-p}) + \circ(|z|^{-p}) = 1$, i.e. $\Re(\alpha z^{-p}) = \circ(|z|^{-p})$.
Now, $\mathcal{H}$ has an horizontal asymptotic branch $\mathcal{D}_1 = \{x \geq, y = 0\}$, and $\mathcal{D}_2 = \{x \geq 0, y = \cos(1)x\}$. When $|z| \to \infty$, on $\mathcal{D}_1$, $z \sim |z|$, and on $\mathcal{D}_2$, $z \sim e^i|z|$. On these two branches, the taylor serie we found implies that :
$\Re(\alpha|z|^{-p}) = |z|^{-p}\Re(\alpha) = \circ(|z|^{-p})$
and $\Re(\alpha(e^i|z|)^{-p}) = |z|^{-p}\Re(\alpha e^{-ip}) = \circ(|z|^{-p})$. This implies $\Re(\alpha) = \Re(\alpha e^{-ip}) = 0$ wich is impossible (since π is irrational). □

Corollary 0.1. *Complex semi-algebraic subsets are not stable under projection.*

Proof. Indeed , otherwise, according to lemma 0.1 , the (real) semi-algebraic subset of C^n would be complex semi-algebraic, but the hyperbola $\mathcal{H}$ gives an ecounter-example. □

Proposition 0.1. *Let $f,\ g \in \mathbb{C}[x]$, such that $\mathbb{R} \subseteq \{|f| = |g|\}$. Then up to multiplication by a scalar, $f(x) = \prod_{k=1\ldots n}(x - a_k)$ and $g(x) = \prod_{k=1\ldots n}(x - b_k)$ where $\forall k$, $b_k \in \{a_k, \overline{a_k}\}$*

Proof. $\forall x \in \mathbb{R}$, $|f(x)|^2 = |g(x)|^2$, but this is now an equality of two real polynomials , which factorize on $\mathbb{R}[X]$ as the product of the $(x - a_k)(x - \overline{a_k})$ (resp. $(x - b_k)(x - \overline{b_k})$), we then conclude by factoriality of $\mathbb{R}[x]$. □

References

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