OBERSEMINAR ARAKELOVTHEORIE: UNIFORM MORDELL

Organizers: Philipp Jell and Florent Martin
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Contents

The Mordell Conjecture, proved by Faltings states the following.

Theorem (Mordell conjecture / Falting's theorem, 1983). *Let X be a curve of genus $g \geq 2$ defined over a number field K. Then $X(K)$ is finite.*

However the following is still a conjecture

Conjecture (Uniform Mordell conjecture). *Let $g \geq 2$ be an integer and let K be a number field. There exists a constant $B(g, K)$ such that for any smooth curve X over K of genus g, $|X(K)| \leq B(g, K)$.*

This conjecture is still open, but by [CHM97, Theorem 1.1], it follows from Lang-Vojta conjecture. Coming back to Mordell conjecture, before Falting's proof, the best result in this direction was the following.

Theorem (Chabauty's theorem, 1941). *Let X be a curve of genus $g \geq 2$ defined over a number field K. Let J be the Jacobian of X, and let r be the Mordell-Weil rank of C which is defined as the rank of the finitely generated abelian group $J(K)$. If $r < g$ then $X(K)$ is finite.*

Of course, Chabauty's theorem is now weaker than Falting's theorem. But Chabauty's ideas, which rely on some p-adic integration theories were further developed by Coleman. This allows Coleman to give explicit bounds in Chabauty's theorem (for simplicity we state it over $\mathbb{Q}$).

Theorem (Coleman's method). *Let X be a curve of genus g over $\mathbb{Q}$, let r be its Mordell-Weil rank and assume as in Chabauty's theorem that $r < g$. Let $p > 2g$ be a prime number where X has good reduction. Then*

$$|X(\mathbb{Q})| \leq |X(\mathbb{F}_p)| + 2g - 2.$$

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Note that by Hasse's theorem, there is a bound for $|X(\mathbb{F}_p)|$, which however depends on p. Thus bound on $|X(\mathbb{Q})|$ is not uniform since the smallest prime of good reduction can be arbitrary high.

Coleman's method was refined by several authors. In 2013, Stoll was able to prove uniform Mordell conjecture for hyperelliptic curves of genus g and Mordell-Weil rank $r \leq g-3$ [Sto13]. The hyperellipticity condition was removed in 2015 by Katz, Rabinoff and Zureick-Brown [KRZ15]:

Theorem. *Let $g \geq 2$ be an integer and let K be a number field. There exists a constant $B(g, K)$ such that for any smooth curve X over K of genus g, and Mordell-Weil rank $r \leq g-3$, $|X(K)| \leq B(g, K)$.*

The goal of this seminar is to explain a proof of this result. On the way we will prove Chabauty's theorem and explain Coleman's method. The methods explained will cover p-adic integration, Berkovich curves and some tropical geometry. We will mainly follow the expository paper [KRZ16], and we also recommend Christian Vilsmeier's master's thesis [Vil16].

1. Abelian varieties (Johannes Sprang/Yigeng Zhao, 18.10.16)

This talk should generally introduce abelian varieties as well as explain some statements about differentials which we will need in later talks.

1.1. **Introduction to abelian varieties.** Introduce the terms *group variety* and *abelian variety* (e.g. [vdGM]). Show that an abelian variety is commutative [HS00, LemmaA.7.1.3]. A nice statement which could also be explained, although it is not needed in later talks, is the fact that over $\mathbb{C}$, any abelian variety is a quotient of $\mathbb{C}^n$ modulo a lattice (e.g. [Mum08, p.2])

The Mordell-Weil Theorem

Theorem 1.1. [HS00, Theorem C.0.1] *Let A be an abelian variety over a number field K. Then $A(K)$ is a finitely generated abelian group.*

should be stated and explained.

1.2. **Differentials on abelian varieties.** Define (degree 1) differentials on an abelian variety and explain what it means for them to be invariant and closed. Show that all global differentials on an abelian variety are invariant and that all invariant differentials are closed:

Proposition 1.2.

$$\Omega^1_{A/K}(A) = \Omega^1_{inv}(A) = Z^1_{DR}(A)$$

For the first equality, once can use [vdGM, Proposition 1.5] (here you only need that A is a proper group variety). For the second one can use [Sha13, Chapter VIII, Section 1.3. Lemma] (here you only need that A is commutative).

2. Jacobians (Johann Haas, 25.10.16)

Define Jacobians, e.g. following the exposition in [HS00, A8]. Explain the group law, that Jac_C is of dimension $g(C)$ and how to get maps $i : C \to \mathrm{Jac}_C$ (called Abel Jacobi maps). Possible things to explain would be the classical construction of Jac_C over $\mathbb{C}$ using integration along cycles (e.g. [HS00, A7]) or the moduli approach (e.g. [Mil86, p.2-3]).

Define the Mordell-Weil rank of a curve:

Definition 2.1. Let C be a curve over a number field K. The Mordell-Weil rank of C is $\mathrm{rank}(\mathrm{Jac}_C(K))$.

Explain that Proposition 1.2 implies that there is an isomorphism $H^0(C, \Omega^1) \simeq H^0(\mathrm{Jac}_C, \Omega^1)$ and state that one of these is given by pullback along any Abel- Jacobi map (Proposition 2.2 from [Mil86]).

3. Uniformization of abelian varieties over non-archimedean fields (Thomas Fenzl, 08.11.16)

One should review the work of Raynaud and Bosch-Lütkebohmert on uniformization of abelian varieties over non-archimedean fields, as well as Abel-Jacobi maps in this context following section 4 of [BR15].

4. p-adic Abelian integrals (by Stefan Stadloeder, 15.11.16)

The goal of this talk is to cover the content of [KRZ16, § 4.1] (which is a more detailed version of [KRZ15, § 3.3]). It would be good to make a clear distinction between $\mathbb{C}_p$-Lie groups and $\mathbb{C}_p$-analytic spaces.

4.1. **p-adic logarithm.** The goal is to prove:

Theorem 4.1. *Let K be a complete subfield of $\mathbb{C}_p$, let A be an abelian variety over K. Then there exists a unique homomorphism of K-Lie groups*

$$\log_A : A(K) \to \mathrm{Lie}(A)$$

such that $d\log_A : \mathrm{Lie}(A) \to \mathrm{Lie}(A)$ is the identity.

To prove this theorem the first step could be to prove:

Lemma 4.2. *Let K be a complete subfield of $\mathbb{C}_p$. Let A be an abelian variety of K and let $x \in A(K)$. Then there exists a sequence of positive integers $(n_i)_{i\in\mathbb{N}}$ such that $n_i x \xrightarrow[i\to+\infty]{} 0$.*

For the proof of this lemma, one can assume that $K = \mathbb{C}_p$ and follow the arguments of [Col89, Theorem 4.1] (or equivalently see [Col85, Lemma 2.10] and the remark following).

The second step of the proof of Theorem 4.1 can be done following [Bou98, III 7.6, Proposition 10 and 11] (having in mind that Lemma 4.2 implies that $A(K)_f = A(K)$ with the notations of [Bou98, III 7.6]).

It would be good to stress that $\log_A$ is a morphism of K-Lie groups which does not come from a morphism of K-analytic spaces $A^{\mathrm{an}} \to \mathrm{Lie}(A)^{\mathrm{an}}$.

4.2. **p-adic integrals on abelian varieties.** Let K be a complete subfield of $\mathbb{C}_p$ and let A be an abelian variety of K. For $\omega \in H^0(A, \Omega^1)$ and $P \in A(K)$, define the *abelian integral* ${}^{\mathrm{Ab}}\int_0^P \omega \in K$. List some properties, in particular:

Proposition 4.3. *(1) For each $\omega \in H^0(A, \Omega^1)$, $P \mapsto {}^{\mathrm{Ab}}\int_0^P \omega \in K$ is a morphism of K-Lie groups.*

(2) If A has good reduction, and $\pi : A^{an} \to \tilde{A}$ is the reduction map, then ${}^{\mathrm{Ab}}\int$ is given by antiderivation on the residue class $\pi^{-1}(1)$ which is isomorphic to an open K-analytic ball of radius 1 and of dimension $\dim(A)$ *(see* [KRZ16, 4.1.1]*).*

4.3. **p-adic integrals on curves.** Explain abelian integrals on curves following [KRZ16, 4.1.3]. One should explain the properties (1)–(4) in [KRZ16, 4.1.3], and in particular the property (4).

5. Chabauty's theorem (by Timo Keller (?), 22.11.16)

As mentioned in the introduction, for over 40 years, the following was the best known result in the direction of the Mordell conjecture

Theorem 5.1. *Let C be a curve over a number field K with Mordell-Weil rank $r < g$. Then $C(K)$ is finite.*

Prove this result. For the presentation follow § 5.1 in [Ser97].

6. Coleman's method (Helene Sigloch, 29.11.16)

The goal of this talk is to cover [MP12, § 5], hence to explain the proof of the following result due to Coleman [Col85].

Theorem 6.1. *Let X be a curve of genus g over $\mathbb{Q}$, let r be its Mordell-Weil rank and assume as in Chabauty's theorem that $r < g$. Let $p > 2g$ be a prime number where X has good reduction. This means that there exists $\mathcal{X}$ a smooth projective scheme over $\mathbb{Z}_p$ which is a model of $X_{\mathbb{Q}_p}$. Then*

$$|X(\mathbb{Q})| \leq |\mathcal{X}(\mathbb{F}_p)| + 2g - 2.$$

One should follow the proof in [MP12, § 5]. In particular one should prove p-adic Rolle's lemma on open discs (see [MP12, Lemma 5.1]).

One should also explain why this theorem implies Chabauty's Theorem. If there is time, one cold give some examples of § 8 of [MP12] and/or explain the version of the theorem on number fields ([Col85, statement (ii) in the introduction]).

7. Berkovich curves and their skeletons (by Martino Stoffel, 06.12.16)

One should explain 3.1–3.3 in [KRZ16] which is an exposition of non-archimedean curves and their skeletons.

8. Potential theory on Berkovich curves and metrics on the cotangent sheaf (by Veronika Wanner, 13.12.16)

8.1. **Potential theory on Berkovich curves.** Introduce potential theory on Berkovich curves following 3.4 [KRZ16].

8.2. **Formal metrics on the cotangent sheaf.** Explain how a semistable model $\mathcal{X}$ of a curve X induces a formal metric on $\Omega^1_{X/K}$ following [KRZ16, 3.5] and [KRZ15, 2.4].

For the presentation of these reults, the more general (and visibly intrinsic) approach to metrizing the cotangent sheaf by Temkin [Tem] could be helpful.

9. Berkovich-Coleman Integration (Klaus Künnemann, 20.12.16)

Definition 9.1. [KRZ16, Definition 4.2.2.] The *Berkovich-Coleman integration theory* is the unique pairing

$$^{BC}\int : \mathcal{P}(Y) \times Z^1_{dR}(Y) \to \mathbb{C}_p$$

which is $\mathbb{C}_p$ linear in ω, concatenation-linear in the path γ, depends only on the fixed end points homotopy class of $\gamma \in \mathcal{P}(Y)$, satisfies a projection formula, fundamental theorem of calculus + some normalization.

Explain this axioms and the role of the choice of a 'branch of logarithm'. You do not have to go into the explicit construction of this pairing

Also explain the special case of integration on totally degenerate Jacobians [KRZ16, Section 4.2.3] and Berkovich Coleman integration on a curve [KRZ16, Section 4.2.3].

10. Comparing the integrals (10.01.17)

The goal of this talk is to compare the two integrations theories $^{\mathrm{Ab}}\int$ and $^{BC}\int$. On should cover the section 4.3 of [KRZ16] (it is not necessary to mention [KRZ16, Corollary 4.3.13]).

11. Stoll decomposition (by Christian Vilsmeier, 17.01.17)

The goal of this talk is to explain the result of M. Stoll [Sto13, Proposition 5.3] (this is quoted as Proposition 4.5 in [KRZ16]). Namely, one has to prove that if X a smooth projective curve over $\mathbb{Q}_p$ of genus $g \geq 2$, then one can cover $X(\mathbb{Q}_p)$ by some open discs and annuli whose number is bounded in terms of g. One should follow sections 4 and 5 of [Sto13] and a there is also a brief discussion of it in [KRZ16, § 4.4].

12. p-adic Rolle's theorem on an annulus (24.01.17)

The key theorem we need for the uniform bound is the following:

Theorem 12.1. [KRZ15, Corollary 4.18] [KRZ16, Theorem 5.1.6] *Let $\omega \in H^0(X, \Omega^1_{X/\mathbb{C}_p})$ be a nonzero global differential and suppose that ω is exact on an open annulus U, i.e. $w = df$ for an analytic function f on U. Then f has at most $2N_p(r, 2g-2)$ zeros on U_r.*

Explain this result, using the exposition in [KRZ16, Section 5.1.] and the preliminary results, which are used in the proof of [KRZ15, Theorem 4.17].

You can focus on the case of annuli, since this is all we need for the prove of the Uniform Mordell Weil conjecture.

13. Conclusion (31.01.17)

Prove the main result

Theorem 13.1. *Let $g \geq 2$ be an integer and let K be a number field. There exists a constant $B(g, K)$ such that for any smooth curve X over K of genus g, and Mordell-Weil rank $r \leq g - 3$, $|X(K)| \leq B(g, K)$.*

This is done in 5.2.1 and 5.2.3 in [KRZ16] and 5.1 [KRZ15]. Since these are quite short, it would be good if the preliminary results which are used are recalled.

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