Introduction to Berkovich spaces

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1 Naive non-Archimedean manifolds

A field k is non-Archimedean if it is equipped with a complete non-Archimedean norm $|\cdot| : k \to \mathbb{R}_+$. This means:

1. $|x+y| \leq \max\{|x|, |y|\}$.
2. $|xy| = |x| \cdot |y|$.
3. $|x| = 0$ if and only if $x = 0$.
4. $(k, |\cdot|)$ is complete.

The norm $|\cdot|$ induces a metric on k. We set

$$k^\circ = \{x \in k \mid |x| \leq 1\} \qquad k^{\circ\circ} = \{x \in k \mid |x| < 1\} \qquad \tilde{k} = k^\circ / k^{\circ\circ}$$

Due to the ultrametric inequality, k° is open and closed. Examples of non-Archimedean fields are $\mathbb{Q}_p, \mathbb{F}_q((t)), \mathbb{C}((t)), \mathbb{C}_p := \widehat{\mathbb{Q}_p^{\mathrm{alg}}}$, any field k with the trivial norm ($|x| = 1$ if $x \neq 0$ and $|0| = 0$)... Note that $\mathbb{C}_p$ is algebraically closed.

Like on $\mathbb{R}$ and $\mathbb{C}$, there are power series on k. For instance:

1. $\frac{1}{1-X} = \sum_{n \geq 0} X^n$ converges for $|X| < 1$.
2. $\exp(X) = \sum_{n \geq 0} \frac{X^n}{n!}$ converges for $\begin{cases} |X| < \frac{1}{p^{\frac{1}{p-1}}} & \text{in characteristic } (0,p), ex : \mathbb{Q}_p. \\ |X| < 1 & \text{in characteristic}(0,0), ex. \mathbb{C}((t)). \end{cases}$

We will say that a map between two open subsets $U \subset k^m$, $V \subset k^n$ is *naively holomorphic* if it is locally given by convergent power series. We define *naive k-analytic manifolds* as topological spaces obtained by gluing open subsets of k^n along biholomorphic maps. A map between naive k-analytic manifolds is a *naive holomorphic map* if it is given locally by naive holomorphic maps between the open subsets of the charts.

This definition is pathological for the following reasons.

1. [Ser65] If k is a local field ($\mathbb{F}_q((t))$ or a finite extension of $\mathbb{Q}_p$), and $d > 0$ is a positive integer, there are only $q-1$ (where $q = \mathrm{Card}(\tilde{k})$) isomorphism classes of compact naive k-analytic manifolds of a given dimension d:

$$\{(k^\circ)^d, (k^\circ)^d \coprod (k^\circ)^d, \ldots, \coprod_{q-1} (k^\circ)^d\}.$$

2. If k is algebraically closed, and $n > 0$ is an integer, there is no compact naive k-analytic manifold of dimension n. But:

   (a) All nonempty open subsets of k^n are naively biholomorphic.

   (b) If $\mathcal{X}$ is a smooth k-algebraic variety of dimension n, the naive k-analytic manifold $\mathcal{X}^{\text{naive-an}}$ is biholomorphic to k^n.

What is the problem? There are too many open subsets, and k is totally disconnected. The closed unit disc is the disjoint union of its open unit balls:

$$k^\circ = \coprod_{\lambda \in \tilde{k}} \{x \in k^\circ \mid \tilde{x} = \lambda\}.$$

For instance

$$\mathbb{Z}_p = \coprod_{i=1\ldots p-1} i + p\mathbb{Z}_p.$$

If $g : \tilde{k} \to k$ is any function, then $f : k^\circ \to k$ defined by $x \in k^\circ \mapsto g(\tilde{x})$ is a locally constant naive holomorphic function on k° which is not constant. In particular, f is not given by a power series converging on k°.

What is the solution? There are several approaches.

1. The theory of rigid spaces (Tate): the idea is to restrict to a class of admissible covers.

2. Add new points to the spaces: Berkovich spaces, adic spaces.

3. Use formal schemes over k° and inverse formal admissible blowing-ups (Raynaud).

4. ...

As in algebraic geometry, these approaches always consider points in the algebraic closure.

2 Affinoid algebras and spaces

2.1 Definitions

The Tate-algebra in n variables is

$$k\{T_1, \ldots, T_n\} := \{\sum_{\nu \in \mathbb{N}^n} a_\nu T^\nu \mid a_\nu \in k,\ a_\nu \xrightarrow[\nu_1 + \ldots \nu_n \to \infty]{} 0\}.$$

These are the power series which converge on the closed unit polydisc. It is a UFD, a Noetherian and regular ring... We equip $k\{T_1, \ldots, T_n\}$ with a k-Banach algebra norm as as follows:

$$\|\sum_{\nu \in \mathbb{N}^n} a_\nu T^\nu\| = \max_\nu |a_\nu|.$$

If I is an ideal of $k\{T_1, \ldots, T_n\}$, we equip the quotient $\mathcal{A} := k\{T_1, \ldots, T_n\}/I$ with the residue norm: for $f \in \mathcal{A}$,

$$\|f\|_{\mathcal{A}} := \inf\{\|g\| \mid g \in k\{T_1, \ldots, T_n\},\ f = g + I\}.$$

One can check that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a k-Banach algebra (this follows from the fact that ideals are closed in $k\{T_1, \ldots, T_n\}$). $\mathcal{A}$ is called a strictly k-affinoid algebra.

More generally, we set

$$k\{r_1^{-1}T_1\dots,r_n^{-1}T_n\} := \{\sum_{\nu\in\mathbb{N}^n} a_\nu T^\nu \mid a_\nu \in k,\ |a_\nu| r^\nu \xrightarrow[\nu_1+\dots\nu_n\to\infty]{} 0\}.$$

These are the power series which converge on the closed polydisc of polyradius $(r_1,\dots,r_n)$. We call quotients $k\{r_1^{-1}T_1\dots,r_n^{-1}T_n\}$ k-affinoid algebras.

A morphism between the k-affinoid algebras $\mathcal{A}$ and $\mathcal{B}$ is a continuous morphism of k-algebras $\mathcal{A}\to\mathcal{B}$.

A bounded multiplicative seminorm on $(\mathcal{A},\|\cdot\|)$ is a map $|\cdot|:\mathcal{A}\to\mathbb{R}_+$ such that

1. $|f+g|\le|f|+|g|$.
2. $|fg|=|f|\cdot|g|$.
3. $|1|=1$.
4. There is a constant $C>0$ such that for all $f\in\mathcal{A}$, $|f|\le C\|f\|$.

Definition 1. *Let $\mathcal{A}$ be a k-affinoid algebra. We set*

$$X=\mathcal{M}(\mathcal{A}) := \{\textit{bounded multiplicative seminorms } |\cdot|:\mathcal{A}\to\mathbb{R}_+\}$$

equipped with the weakest topology making the maps

$$\begin{array}{ccc}\mathcal{M}(\mathcal{A}) & \to\mathbb{R} & \\ |\cdot| & \mapsto & |f|\end{array}$$

continuous for all $f\in\mathcal{A}$. We call X a k-affinoid space.

The k-affinoid spaces will serve as building blocks for k-analytic spaces.

Proposition 2. *$\mathcal{M}(\mathcal{A})$ is a nonempty compact Hausdorff space.*

Notation: we want to consider $X=\mathcal{M}(\mathcal{A})$ as a geometric space and $f\in\mathcal{A}$ as functions on X. So if $x:\mathcal{A}\to\mathbb{R}$ is an element of $\mathcal{M}(\mathcal{A})$, and $f\in\mathcal{A}$, we set

$$|f(x)| := x(f).$$

A basis of open subsets for the topology of $\mathcal{M}(\mathcal{A})$ is given by the sets

$$U=\{x\in\mathcal{M}(\mathcal{A}) \mid \alpha_i<|f_i(x)|<\beta_i,\ i=1\dots n\}$$

where $f_i\in\mathcal{A}$ and $\alpha_i,\beta_i\in\mathbb{R}$.

Let $\varphi:\mathcal{A}\to\mathcal{B}$ be a morphism of k-affinoid algebra. It induces a continuous map $\varphi^*\mathcal{M}(\mathcal{B})\to\mathcal{M}(\mathcal{A})$. If $x:\mathcal{B}\to\mathbb{R}$ is an element of $\mathcal{M}(\mathcal{B})$, $\varphi^*(x)$ is the seminorm

$$f\in\mathcal{A}\mapsto|\varphi(f)(x)|$$

which is obtained as the composition $\mathcal{A}\xrightarrow{\varphi}\mathcal{B}\xrightarrow{x}\mathbb{R}$. A morphism of k-affinoid spaces is a morphism of the form φ^*, so the category of k-affinoid spaces is equivalent to the opposite category of k-affinoid algebras.

Theorem 3. Nullstellensatz for strictly k-affinoid algebras. *If $\mathfrak{m}$ is a maximal ideal of a strictly k-affinoid algebra $\mathcal{A}$, then $\mathcal{A}/\mathfrak{m}$ is a finite extension of k.*

For any finite extension of non-trivially valued non-Archimedean fields $k\to k'$, there exists a unique non-Archimedean norm $|\cdot|_{k'}$ on k' extending the norm of k. So if $\mathfrak{m}$ is a maximal ideal of $\mathcal{A}$, we have a canonical associated norm $|\cdot|_{\mathfrak{m}}$ on $\mathcal{A}/\mathfrak{m}$ and we can associate an element of $\mathcal{M}(\mathcal{A})$

$$\mathcal{A}\to\mathcal{A}/\mathfrak{m}\xrightarrow{|\cdot|_{\mathfrak{m}}}\mathbb{R}.$$

This implies that one has a correspondence between maximal ideals of $k\{T_1,\dots,T_n\}$ and the orbits of $((k^{\mathrm{alg}})^\circ)^n$ modulo $\mathrm{Gal}(k^{\mathrm{alg}}/k)$. More generally, if $\mathcal{A}=k\{T_1,\dots,T_n\}/(f_1,\dots,f_m)$ one has a correspondence between maximal ideals of $\mathcal{A}$ and the set of points $x\in((k^{\mathrm{alg}})^\circ)^n$ such that $f_i(x)=0$ for all i modulo $\mathrm{Gal}(k^{\mathrm{alg}}/k)$.

2.2 Example of the unit disc

Let us assume for this part that k is algebraically closed. We set

$$B = \mathcal{M}(k\{T\})$$

and call it the closed unit disc. If $c \in k^\circ$ and $r \in [0,1]$, we denote the closed disc of center c and radius r by

$$B(c,r) := \{x \in k^\circ \mid |x-c| \leq r\}.$$

We want to classify the points of B.

Type 1. If $c \in k^\circ$ then the multiplicative seminorm

$$f \in k\{T\} \mapsto |f(c)|$$

defines a point of $B = \mathcal{M}(k\{T\})$. These are in correspondence with maximal ideals of $k\{T\}$ and are called points of type (1).

Type 2 and 3. Let us first remark that the Banach norm on $k\{T\}$ given by

$$f = \sum_{n\in\mathbb{N}} a_n T^n \mapsto \|f\| = \max_{n\in\mathbb{N}} |a_n|$$

is multiplicative. In addition $\|f\| = \max_{z\in D(0,1)} |f(z)|$.

More generally if $r \in]0,1]$ let us set

$$|\cdot|_{D(0,r)} : f = \sum a_n T^n \mapsto |f|_{D(0,r)} = \max |a_n| r^n.$$

Then $|\cdot|_{D(0,r)} \in B$, i.e. it is a multiplicative seminorm. In addition, it satisfies for all $f \in k\{T\}$

$$|f|_{D(0,r)} = \max_{z\in D(0,r)} |f(z)|.$$

More generally again, if If $c \in k^\circ$ and $r \in [0,1]$ we define

$$|\cdot|_{D(c,r)} : f = \sum a_n T^n \mapsto |f|_{D(c,r)} = \max |a_n| r^n.$$

Then $|\cdot|_{D(c,r)} \in B$, i.e. it is a multiplicative seminorm. In addition, it satisfies for all $f \in k\{T\}$

$$|f|_{D(c,r)} = \max_{z\in D(c,r)} |f(z)|.$$

When $r \in \sqrt{|k^\times|}$, $|\cdot|_{D(c,r)}$ is called a type of point (2) of B, and when $r \notin \sqrt{|k^\times|}$, $|\cdot|_{D(c,r)}$ is called a point of type (3).

Type 4. The field k is called maximally complete if every family of embedded nonempty discs has a non empty intersection. For instance local fields are maximally complete, but one can prove that $\mathbb{C}_p$ is not maximally complete.

Let us assume that k is not maximally complete, this means that one can find a positive real number $r \in]0,1[$, and for each $\rho \in]r,1]$, an element $c_\rho \in k^\circ$ such that

$$\mathcal{E} := \{D(c_\rho, \rho),\ \rho \in]r,1]\}$$

is a family of embedded discs with empty intersection. We then set

$$|\cdot|_{\mathcal{E}} : f \in k\{T\} \mapsto \inf_{D\in\mathcal{E}} |f|_D.$$

One checks that $|\cdot|_{\mathcal{E}}$ defines a multiplicative seminorm. So $|\cdot|_{\mathcal{E}}$ is an element of B. The assumption that the family $\mathcal{E}$ has empty intersection implies that $|\cdot|_{\mathcal{E}}$ is not of type (1), (2) or (3).

Fact 4. *Any point of $B = \mathcal{M}(k\{T\})$ is of type (1)–(4) (cf. [Ber90, 1.4.4]).*

Make a picture, and draw lines. Check that the map

$$r \in [0,1] \mapsto |\cdot|_{D(c,r)} \in B$$

induces a homeomorphism between $[0,1]$ and its image.

Proposition 5. *$B = \mathcal{M}(k\{T\})$ is arcwise-connected.*

If $\lambda \in k^\circ$, lets us set

$$B^+_{\tilde{\lambda}} = \{x \in B \mid |(T-\lambda)(x)| < 1\}.$$

This depends only on the reduction $\tilde{\lambda} \in \tilde{k}$. Then $B^+_{\tilde{\lambda}}$ are a disjoint family of open subsets of B, but now, $B^+_{\tilde{\lambda}}$ is not closed, and in addition

$$\coprod_{\tilde{\lambda} \in \tilde{k}} B^+_{\tilde{\lambda}} \subsetneq B.$$

Indeed there is just one point missing: $|\cdot|_{D(0,1)}$.

3 Gluing of affinoid spaces

3.1 Example of an annulus

Let $\pi \in k$ such that $0 < |\pi| < 1$ and let us consider the annulus

$$U = \{x \in B = \mathcal{M}(k\{T\}) \mid |\pi| \leq |T(x)| \leq 1\}.$$

Try to make two drawings of U.

We want to see U as a k-affinoid space inside B. What should be the set of analytic functions on U?

$$\mathcal{A}_U = \{f = \sum_{n \in \mathbb{Z}} a_n T^n \mid |a_n| \xrightarrow[n\to\infty]{} 0 \;\; |a_{-n}\pi^{-n}| \xrightarrow[n\to\infty]{} 0\}.$$

If one set $S := \frac{\pi}{T}$,

$$\mathcal{A}_U \simeq k\{S,T\}/(ST-\pi).$$

Fact 6. *The morphism*

$$k\{T\} \to k\{S,T\}/(ST-\pi)$$

induces a homeomorphism between $\mathcal{M}(k\{S,T\}/(ST-\pi))$ and U through the following morphism of k-affinoid spaces

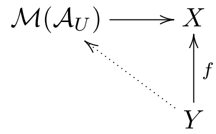
$$\mathcal{M}(k\{S,T\}/(ST-\pi)) \to B.$$

3.2 Affinoid domains

Definition 7. *Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space. A closed subset $U \subset X$ is an affinoid domain of X is there exists a k-affinoid algebra $\mathcal{A}_U$ and a morphism of k-affinoid algebras $\mathcal{A} \to \mathcal{A}_U$ such that*

1. *$\varphi^* : \mathcal{M}(\mathcal{A}_U) \to X$ induces a homeomorphism between $\mathcal{M}(\mathcal{A}_U)$ and U.*

2. *For any morphism of k-affinoid space $f : Y = \mathcal{M}(\mathcal{B}) \to X = \mathcal{M}(\mathcal{A})$ such that $f(Y) \subset U$, there exists a unique morphism of k-affinoid spaces $Y \to \mathcal{M}(\mathcal{A}_U)$ making the following diagram commutative:*



Here is a list of examples.

1. In the example of subsection 3.1, U is an affinoid domain of B.
2. If $f_1, \ldots, f_n \in \mathcal{A}$
$$U = \{x \in X \mid |f_i(x)| \leq 1,\ i = 1 \ldots n\}$$
is an affinoid domain, called a Weierstrass domain, and
$$\mathcal{A}_U \simeq \mathcal{A}\{T_1, \ldots, T_n\}/(f_i - T_i)_{i=1\ldots,n}.$$
3. If $f_1, \ldots, f_n, g \in \mathcal{A}$ generate the unit ideal of $\mathcal{A}$,
$$U = \{x \in X \mid |f_i(x)| \leq |g(x)|,\ i = 1 \ldots n\}$$
is an affinoid domain, called a rational domain, and
$$\mathcal{A}_U \simeq \mathcal{A}\{T_1, \ldots, T_n\}/(f_i - gT_i)_{i=1\ldots,n}.$$

Lemma 8. *If U and V are affinoid domains of X so is $U \cap V$.*

Theorem 9 (Tate's acyclicity Theorem). *Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space and let $U_1, \ldots, U_n$ be a finite cover of X by affinoid domains. Then*
$$\mathcal{A} \to \prod_{i=1\ldots n} \mathcal{A}_{U_i} \to \prod_{1 \leq i < j \leq n} \mathcal{A}_{U_i \cap U_j}$$
is exact.

3.3 Gluing, informal definition

We only sketch the global definition of a k-analytic spaces. For precise definitions, see [Ber93, section 1].

A k-analytic space is a locally Hausdorff topological space X covered by a *nice* family of compact subsets $\{U_i\}$ (the U_i's should be a net of X, cf. [Ber93, section 1]) such that each U_i should be identified with a k-affinoid space, and such that whenever $U_i \subset U_j$, U_i should be identified with an affinoid domain of U_j.

A morphism between two k-analytic spaces is roughly a continuous map $f : X \to Y$ such that whenever $U \subset X$ and $V \subset Y$ are affinoid domains with $f(U) \subset V$, then $f_{|U} : U \to V$ is a morphism of k-affinoid spaces.

4 Analytification and GAGA

4.1 The affine space

We define

$$(\mathbb{A}^n_k)^{\mathrm{an}} = \{\text{multiplicative seminorms } |\cdot| : k[T_1, \dots, T_n] \to \mathbb{R}_+ \text{ extending the norm of } k\}$$

with the weakest topology such that for all $P \in k[T_1, \dots, T_n]$, the map $(\mathbb{A}^n_k)^{\mathrm{an}} \to \mathbb{R}_+$, defined by $|\cdot| \mapsto |P|$ is continuous.

For each $r > 0$, the inclusion $k[T_1, \dots, T_n] \hookrightarrow k\{r^{-1}T_1, \dots, r^{-1}T_n\}$ induces a map

$$\mathcal{M}(k\{r^{-1}T_1, \dots, r^{-1}T_n\}) \to (\mathbb{A}^n_k)^{\mathrm{an}}$$

which induces a homeomorphism between $\mathcal{M}(k\{r^{-1}T_1, \dots, r^{-1}T_n\})$ and its image

$$B^n_r = \{x \in (\mathbb{A}^n_k)^{\mathrm{an}} \mid |T_i(x)| \leq r, \forall i\}.$$

For all $r < s$, B^n_r is an affinoid domain of B^n_s and this equip $(\mathbb{A}^n_k)^{\mathrm{an}}$ with the structure of a k-analytic space.

4.2 Algebraic k-varieties

Let $\mathcal{X}$ be an affine k-variety given by $\mathcal{X} = V(I)$ for some ideal $I \subset k[T_1, \dots, T_n]$. For $r > 0$ we set $X_r := \mathcal{M}(k\{r^{-1}T_1, \dots, r^{-1}T_n\}/I)$. It can be identified with a closed subset of

$$\mathcal{X}^{\mathrm{an}} := \{\text{multiplicative seminorms } |\cdot| : k[T_1, \dots, T_n]/I \to \mathbb{R}_+ \text{ extending the norm of } k\}$$

equipped with the topology inherited from $(\mathbb{A}^n_k)^{\mathrm{an}}$, i.e. the weakest topology such that for all $P \in k[T_1, \dots, T_n]/I$, the map $\mathcal{X}^{\mathrm{an}} \to \mathbb{R}_+$, defined by $|\cdot| \mapsto |P|$ is continuous. For $r < s$, X_r is identified with an affinoid domain of X_s, and this equips $\mathcal{X}^{\mathrm{an}}$ with a structure of k-analytic space.

Example 10. *We have a homeomorphism*

$$(\mathbb{G}^n_{m,k})^{an} \simeq \{\textit{multiplicative seminorms } |\cdot| : k[T_1, T_1^{-1}, \dots, T_n, T_n^{-1}] \to \mathbb{R}_+ \textit{ extending the norm of } k\}.$$

The continuous map

$$\begin{array}{ccc} (\mathbb{G}^n_{m,k})^{an} & \to & \mathbb{R}^n \\ x & \mapsto & (\log |T_i(x)|)_{i=1\dots n} \end{array}$$

is continuous. For $r_1, \dots, r_n \in (\mathbb{R}^*_+)^n$, the seminorm

$$\sum_{\nu \in \mathbb{Z}^n} \mapsto \max_{\nu \in \mathbb{Z}^n} |a_\nu| r_1^{\nu_1} \dots r_n^{\nu_n}$$

is a continuous section of the above map.

When $\mathcal{X}$ is a k-scheme of finite type, one can glue the above construction and define a k-analytic space $\mathcal{X}^{\mathrm{an}}$, which defines a functor $\mathcal{X} \to \mathcal{X}^{\mathrm{an}}$ from the category of k-scheme of locally finite type to the category of k-analytic spaces. As in the complex case one has GAGA-Theorems: for a k-scheme of locally finite type $\mathcal{X}$ there is a functor $F \mapsto F^{\mathrm{an}}$ from coherent $\mathcal{O}_{\mathcal{X}}$-modules to coherent $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}$-modules such that

1. if $\varphi : \mathcal{Y} \to \mathcal{X}$ is a proper morphism of schemes of locally finite type

$$(R^p \varphi_* F)^{\mathrm{an}} \simeq R^p \varphi^{\mathrm{an}}_* (F^{\mathrm{an}}).$$

2. If $\mathcal{X}$ is proper, the functor $F \to F^{\mathrm{an}}$ is an equivalence of categories.

5 Topological properties

Let X be k-analytic space. Then X is

1. locally compact.
2. locally arcwise connected.
3. If X has a strictly semi-stable model or is the analytification of a quasi-projective variety, X retracts on a finite CW-complex.

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